

# Nondegeneracy and Stability of Antiperiodic Bound States for Fractional Nonlinear Schrödinger Equations

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## Abstract

We consider the existence and stability of real-valued, spatially antiperiodic standing wave solutions to a family of nonlinear Schrödinger equations with fractional dispersion and power-law nonlinearity. As a key technical result, we demonstrate that the associated linearized operator is nondegenerate when restricted to antiperiodic perturbations, i.e. that its kernel is generated by the translational and gauge symmetries of the governing evolution equation. In the process, we provide a characterization of the antiperiodic ground state eigenfunctions for linear fractional Schrödinger operators on  $\mathbb{R}$  with real-valued, periodic potentials as well as a Sturm-Liouville type oscillation theory for the higher antiperiodic eigenfunctions.

## 1 Introduction

In this paper, we consider the existence and stability properties of real-valued, spatially periodic solutions to a class of fractional nonlinear Schrödinger equations (fNLS) of the form

$$(1.1) \quad iu_t - \Lambda^\alpha u + \gamma|u|^{2\sigma}u = 0, \quad (x, t) \in \mathbb{R}^2.$$

where subscripts denote partial differentiation. Here and throughout,  $u = u(x, t)$  is a generally complex-valued function, and the pseudodifferential operator  $\Lambda := \sqrt{-\partial_x^2}$ , referred to as Calderon's operator, is of first order and, acting on  $2T$ -periodic functions, is defined by its Fourier multiplier via

$$\widehat{\Lambda f}(n) = \frac{\pi|n|}{T} \hat{f}(n), \quad n \in \mathbb{Z}.$$

Further,  $\gamma = \pm 1$  distinguishes between focusing (attracting)  $\gamma = +1$  and defocusing (repulsive)  $\gamma = -1$  nonlinearities.

The parameter  $\alpha \in (0, 2]$  describes the fractional dispersive nature of the equation. When  $\alpha = 2$ , the operator  $\Lambda^2 = -\partial_x^2$  denotes the local (positive) Laplacian. In this classical case, (1.1) reduces to the well-studied nonlinear Schrödinger equation (NLS), which is known to serve as canonical model describing weakly nonlinear wave propagation in dispersive media; see, for example, [49]. When  $\alpha \in (0, 2)$ ,  $\Lambda^\alpha$  denotes the so-called *fractional Laplacian*, which arises naturally in a

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variety of applications including the continuum limit of discrete models with long range interaction [37], dislocation dynamics in crystals [12], mathematical biology [43], water wave dynamics [31], and financial mathematics [14]; see also [10] for a recent discussion on applications. For such  $\alpha$ , the *nonlocal* fNLS (1.1) has been introduced by Laskin [38] in the context of fractional quantum mechanics, in which one generalizes the standard Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. A rigorous derivation in the context of charge transport in bio polymers (like DNA), can be found in [37]. Finally, we point out the case  $\alpha = 1$  may be thought to describe the relativistic dispersion relation  $\omega(\xi) = \sqrt{|\xi|^2 + m^2}$ , an observation recently utilized in the mathematical description of Boson-stars; see [19].

Throughout our analysis, we will be concerned with solutions of the form

$$(1.2) \quad u(x, t) = e^{i\omega t} \phi(x - ct; c),$$

where  $\omega, c \in \mathbb{R}$  are parameters  $\phi$  is a bounded solution to the (generally) nonlocal profile equation

$$\Lambda^\alpha \phi + \omega \phi + ic\phi' - \gamma|\phi|^{2\sigma}\phi = 0.$$

When  $c = 0$ , the focusing fNLS is well-known to admit standing solitary waves that are asymptotic to zero at spatial infinity. Among such solitary wave solutions, specific attention is often paid to the positive, radially symmetric solutions typically referred to as “ground states”. The stability of such ground states dates back to the work of Cazenave and Lions [13] and Weinstein [50, 51] on the classical case  $\alpha = 0$ , where the authors utilize the method of concentration compactness along with the construction of appropriate Lyapunov functionals. For  $\alpha \in (1, 2]$ , such ground states are known to be orbitally stable provided the nonlinearity is energy sub-critical, i.e. if  $0 < \sigma < \alpha$ ; see [27, 51], for example. While no nontrivial localized solutions exist in the defocusing case  $\gamma = -1$ , it is known in the classical case  $\alpha = 2$  to admit so-called black solitons of the form (1.2) corresponding to monotone front-like solutions asymptotic to constants as  $x \rightarrow \pm\infty$ . The dynamics and stability of black solitons has been studied in numerous works; see, for example, [4, 23, 24]. For the fractional case, the authors plan to report on the existence, nondegeneracy, and stability of black solitons in the defocusing case in a future work.

The stability of periodic standing waves of (1.1) is considerably less understood than their asymptotically constant counterparts, even in the classical  $\alpha = 2$  case. Recall that when  $\alpha = 2$  and  $\sigma = 1$ , Rowlands’ [48] formally demonstrated the spectral instability to long-wavelength (i.e. modulational) perturbations of all periodic standing waves in the focusing case, along with the stability (to long-wavelength perturbations) of such waves in the defocusing case. Rowlands’ results were later rigorously established by Gallay and Haragus [21] for small amplitude waves, and by Gustafson, Le Coz, and Tsai [28] and Deconinck and Segal [15] for arbitrary amplitude waves. The spectral stability to arbitrary bounded perturbations of periodic standing waves in the defocusing NLS ( $\alpha = 2$ ) was later shown by Gallay and Haragus [21] for small amplitude waves and by Bottman, Deconinck, and Nivala [7], again using complete integrability, for waves of arbitrary amplitude. These observations motivate us to restrict much of our attention to the defocusing case  $\gamma = -1$ , as we expect such waves in the focusing case to be modulationally unstable for general  $\alpha$ , although, this has not been verified. In fact, to the authors’ knowledge there has been no rigorous study into the dynamics of such waves in the fractional case.

In this work, we will be concerned with the *nonlinear stability* of periodic standing waves of (1.1) in the genuinely nonlocal case  $\alpha \in (0, 2)$ . We mark that since the governing evolution equation

is invariant under phase rotation and spatial translation, i.e. the map

$$u(x, t) \mapsto e^{i\beta} u(x - x_0, t), \quad x_0, \beta \in \mathbb{R}$$

preserves the class of solutions of (1.1), we should only expect stability up to these invariances. Such *orbital stability* results for solutions of (1.1) have been obtained in the local case  $\alpha = 2$ . The first result in this direction we are aware of in the periodic case was due to Angulo-Pava [2], where, in the focusing case with  $\alpha = 2$ , the orbital stability of dnoidal type (hence strictly positive) standing waves was established to perturbations with the same period as the underlying wave. Pava's analysis relied on a direct adaptation of the classical approach to orbital stability by Grillakis, Shatah, and Strauss [25, 26] and could not be extended to cnoidal type (sign changing) solutions in either the focusing or defocusing cases. This issue was later resolved by Gallay and Haragus [20], where the authors demonstrated the stability of cnoidal waves in the defocusing cubic NLS ( $\alpha = 2$ ,  $\sigma = 1$ ) to perturbations with the same period as the *modulus* of the underlying wave. This restrictive class of perturbations is essential to employ the techniques of [25, 26] since, as noted by Pava, the Hessian of the associated Lagrangian at such a cnoidal wave has two negative eigenvalues when acting on co-periodic functions, invalidating the structural hypotheses of [25, 26]. See, however, the recent work [22] where the restriction to antiperiodic perturbations is removed through the use of the completely integrable structure of the cubic, defocusing NLS.

Here, we take matters further and study the existence and nonlinear stability of periodic standing waves of the fNLS (1.1) with fractional dispersion  $\alpha \in (0, 2)$ , provided, following [20], we appropriately restrict the class of perturbations. In the defocusing case, where we primarily restrict our attention, we will show in Section 2 that, for each  $T > 0$ , there exists a three-parameter family of real-valued,  $T$ -antiperiodic standing waves, i.e.  $2T$ -periodic waves with

$$\phi(x + T) = -\phi(x),$$

arising as local minima of the Hamiltonian energy subject to conservation of momentum: see Proposition 2.1 and Lemma 2.2. Once existence is established, our main goal is to establish the nonlinear stability of such real-valued,  $T$ -antiperiodic standing waves to small  $T$ -antiperiodic perturbations: see Theorem 4.1.

A key step in our stability analysis is to show that the Hessian of the Hamiltonian energy is *nondegenerate* at such an antiperiodic, local constrained minimizer of the defocusing fNLS; that is, that the kernel is generated only by spatial translations and phase rotations. The nondegeneracy of the linearization is known to play an important role in the stability of traveling and standing waves (see [51], [40], and [20]) and in the blowup analysis (see [36] [49], for instance) of the related dynamical equation. In the case of the classical NLS with cubic nonlinearity, the nondegeneracy at such antiperiodic standing wave solutions was established by Gallay and Haragus [20, Proposition 3.2]. Their proof, however, fundamentally relies on ODE techniques, in particular on the Sturm-Liouville theory for ODEs and a-priori bounds on the number of linearly independent solutions to the linearized equations, and is hence not directly applicable to the nonlocal case  $\alpha \in (0, 2)$ . Nevertheless, Frank and Lenzmann [18] recently established the nondegeneracy of *solitary* waves for a family of nonlocal evolution equations, including the focusing fNLS. Their analysis relied on the development of a suitable substitute for the Sturm-Liouville theory, following from the characterization of the fractional Laplacian as a Dirichlet-to-Neumann operator for a *local* elliptic problem in the upper half-plane, allowing them to bound from above the number of sign changes of eigenfunctions for fractional linear Schrödinger operators on the line. This oscillation theory was

recently extended to the periodic setting in [42], where the authors considered the orbital stability of periodic traveling waves of the fractional gKdV equation.

While the oscillation theory in [42] seems to apply directly to periodic standing waves in the focusing fNLS, see Remark 5.4, it requires considerable modification in the defocusing case, accounting for the  $T$ -antiperiodicity of the eigenfunctions compared with the  $T$ -periodicity of the potential in the associated linear Schrödinger operators. We point out that even in the classical  $\alpha = 2$  case, antiperiodic ground states for linear Schrödinger operators *need not be simple*. Indeed, it is not difficult to cook up examples of potentials for which the associated Schrödinger operator will have an antiperiodic ground state with multiplicity two; see [41] for instance. We handle this difficulty by restricting such operators to even and odd antiperiodic subspaces, demonstrating that the associated linear semigroup is positivity improving on these individual subspaces. A twist on standard Perron-Frobenius arguments then yields a characterization of the antiperiodic ground states of such fractional linear Schrödinger operators restricted to even and odd functions: see Theorem 3.9. Further, using antiperiodic rearrangement inequalities developed in Appendix A, we demonstrate that the ordering of the even and odd antiperiodic ground states for a fractional linear Schrödinger operator depends explicitly on the monotonicity properties of the real-valued periodic potential: see Proposition 3.10. Once the appropriate ground state theories are developed, then provide a suitable oscillation theory for higher antiperiodic eigenfunctions by following the arguments in [18, 42]: see Lemma 3.11.

We emphasize that the realness of the  $T$ -antiperiodic solutions  $\phi$  discussed above is *absolutely crucial* to our analysis, guaranteeing that the Hessian of the Lagrangian functional encoding solutions of the profile equation as critical points acts as a diagonal operator on  $L_a^2(0, T)$ . This diagonal property reduces the nondegeneracy analysis to the study of two scalar linear fractional Schrödinger operators, which is precisely the setting where our techniques from Section 3 apply. In the “nontrivial phase” case, corresponding to genuinely complex-valued solutions  $\phi$ , the Hessian operator couples the real and imaginary parts of the perturbations, invalidating the strategy and techniques contained in this paper: see equation (3.2) below. The corresponding nondegeneracy and stability analysis for the nontrivial phase solutions is completely open in the nonlocal case and is a *very* interesting direction for future research. To our knowledge, the only nondegeneracy result known in the nontrivial phase case was provided by in [20] using, as discussed before, restrictive ODE techniques.

In the focusing case, we construct real-valued antiperiodic solutions through a different constrained minimization procedure, producing solutions that may or may not be constrained energy minimizers: see Proposition 5.1. By applying our techniques developed for the defocusing case, we establish in Proposition 5.3 the nondegeneracy of all of these waves *independent* of whether they are constrained energy minimizers. As an application, we then characterize the stability of these waves in terms of variations of the  $L^2$  norm with respect to an appropriate parameter: see Theorem 5.6 and Proposition 5.7.

The outline of the paper is as follows. In Section 2 we use variational arguments to establish the existence of antiperiodic solutions of the profile equation (2.1) in the defocusing case. We then consider the nondegeneracy of these antiperiodic solutions in Section 3, followed by a proof of their orbital stability in Section 4. Finally, in Section 5 we discuss an extension of our work to the focusing case. Appendix A contains proofs of the relevant rearrangement inequalities used in the development of the antiperiodic ground state theory used in Section 3.

## 2 Existence of Constrained Local Minimizers in Defocusing Case

We begin our analysis by establishing the existence of periodic waves of the form (1.2) of the *defocusing* ( $\gamma = -1$ ) fNLS (1.1). Substituting the standing wave ansatz (1.2) into (1.1) yields the nonlocal *profile equation*

$$(2.1) \quad \Lambda^\alpha \phi + \omega \phi + ic\phi' + |\phi|^{2\sigma} \phi = 0, \quad \omega, c \in \mathbb{R}.$$

where here  $\phi$  is generally a complex-valued function and  $\sigma > 0$ ; further restrictions will be placed on  $c$ ,  $\omega$ , and  $\sigma$  later. Here and throughout, given a finite period  $T > 0$  we consider for each  $\alpha > 0$  the operator  $\Lambda^\alpha$  as a closed operator on

$$L^2_{\text{per}}([0, 2T]; \mathbb{C}) := \{f \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{C}) : f(x + 2T) = f(x) \quad \forall x \in \mathbb{R}\}$$

with dense domain  $H^\alpha_{\text{per}}([0, 2T]; \mathbb{C})$ , defined via its Fourier series as

$$\Lambda^\alpha f(x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left| \frac{\pi n}{T} \right|^\alpha e^{\pi i n x / T} \hat{f}(n), \quad \alpha \geq 0.$$

We will be interested primarily in *real-valued, standing* wave solutions of (1.1), in which case  $c = 0$ . To motivate the expected structure of such solutions, we note that when  $\alpha = 2$  and  $c = 0$  the profile equation (2.1) is integrable and, upon integration, can be expressed as

$$\frac{1}{2} (\phi')^2 = H - V(\phi; \omega)$$

where  $H \in \mathbb{R}$  denotes the ODE energy and

$$V(\phi; \omega) := -\frac{\omega}{2} \phi^2 - \frac{1}{2\sigma + 2} \phi^{2\sigma + 2}$$

denotes the effective potential energy. Observe that the potential  $V$  is even for every  $\omega \in \mathbb{R}$  and, possesses a unique local minimum when  $\omega < 0$ , yielding the existence of a one-parameter family, parameterized by  $H$ , of periodic orbits<sup>1</sup> oscillating symmetrically about the equilibrium solution  $\phi = 0$ ; see Figure 1. Further, up to translations these waves can be chosen to be even and *antiperiodic*, i.e. they satisfy

$$\phi(x + T) = -\phi(x)$$

where  $2T > 0$  denotes the fundamental period of  $\phi$ . Note that while such solutions can be expressed explicitly in terms of the Jacobi elliptic function  $\text{cn}$ , the authors are unaware of such an explicit solution formula for  $\alpha \in (0, 2)$ . Nevertheless, for each  $\alpha \in (1, 2)$  and  $T > 0$  we expect to be able to construct a three-parameter family of real-valued,  $T$ -antiperiodic solutions of the profile equation (2.1).

In the absence of the integrable ODE structure present in the case  $\alpha = 2$ , we will construct real-valued, antiperiodic solutions of (2.1) with  $c = 0$  through a constrained minimization argument. We emphasize, however, that in our forthcoming stability analysis it will be important that our

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<sup>1</sup>When  $\omega > 0$ , the potential  $V$  is strictly decreasing on  $(0, \infty)$  and hence no nontrivial bounded solutions exist in this case.

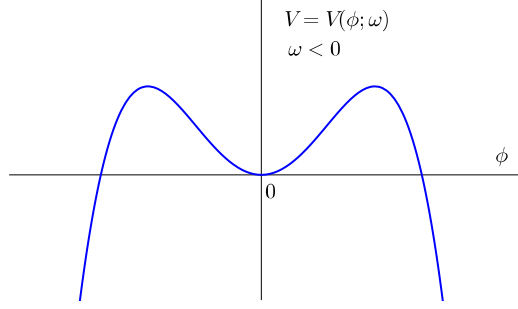


Figure 1: The effective potential  $V(\phi; \omega)$  for the defocusing NLS,  $\omega < 0$ .

real-valued antiperiodic solutions be (locally) embedded into a larger family of *complex-valued, traveling* waves with  $c \neq 0$ ; see Remark 3.13 below. In the local case  $\alpha = 2$ , this embedding is easily handled due to the existence of an exact Galilean invariance: precisely, if  $u(x, t)$  is a solution of (1.1) with  $\alpha = 2$ , then so is

$$(2.2) \quad \mathcal{G}_c u(x, t) = e^{i\frac{c}{2}x - i\frac{c^2}{4}t} u(x - ct, t)$$

for each wave speed  $c \neq 0$ . However, when  $\alpha \neq 2$  no such exact Galilean invariance exists. As a result, our variational arguments below will actually construct a general class of antiperiodic traveling wave solutions of (1.1) of the form (1.2) with  $|c|$  sufficiently small, and then show through various rearrangement arguments that when  $c = 0$  the resulting waves can be chosen to be real-valued with appropriate dependencies on the wave speed.

To this end, given a  $T > 0$  we work throughout in the  $L^2$ -based Lebesgue and Sobolev spaces over the antiperiodic interval  $[0, T]$ . Define the *real* vector space

$$L_a^2([0, T]; \mathbb{C}) := \{f \in L_{\text{loc}}^2(\mathbb{R}; \mathbb{C}) : f(x + T) = -f(x) \quad \forall x \in \mathbb{R}\}$$

equipped with inner product  $\langle u, v \rangle = \text{Re} \int_0^T u \bar{v} \, dx$ , and for each  $\alpha \in (0, 2)$  define

$$(2.3) \quad H_a^{\alpha/2}([0, T]; \mathbb{C}) := \left\{ f \in H_{\text{loc}}^{\alpha/2}(\mathbb{R}; \mathbb{C}) : f \in L_a^2([0, T]; \mathbb{C}) \right\}$$

considered as a *real* vector space with inner product

$$(u, v) := \text{Re} \int_0^T (u \bar{v} + \Lambda^{\alpha/2} u \overline{\Lambda^{\alpha/2} v}) \, dx.$$

By Sobolev embedding, it is trivial to see that  $H_a^{\alpha/2}(0, T)$  is a closed subspace of  $H_{\text{per}}^{\alpha/2}(0, T)$ , and hence is itself a Hilbert space, for all  $\alpha > 1$ . Furthermore, we identify the dual space  $H_a^{\alpha/2}(0, T)^*$  with  $H_a^{-\alpha/2}(0, T)$  via the pairing

$$(2.4) \quad \langle u, v \rangle := \text{Re} \int_0^T u \bar{v} \, dx, \quad u \in H_a^{\alpha/2}(0, T)^*, \quad v \in H_a^{\alpha/2}(0, T).$$

Throughout, unless otherwise stated, we will use the slight abuse of notation that

$$H_*^s(0, T) := H_*^s([0, T]; \mathbb{C})$$

where here  $*$  stands for any of “per”, or “a”. We note that we may at times work with the subspace of *real-valued* functions  $H_a^{\alpha/2}([0, T]; \mathbb{R})$  in  $H_{\text{per}}^{\alpha/2}(0, T)$ ; when the choice of scalar field is irrelevant or obvious from context, we will simply write  $H_a^{\alpha/2}(0, T)$  for these spaces.

To begin with our existence theory, we consider  $\alpha > 1$  and define the functionals

$$K(u) := \frac{1}{2} \int_0^T |\Lambda^{\alpha/2} u|^2 dx, \quad P(u) := \frac{1}{2\sigma + 2} \int_0^T |u|^{2\sigma+2} dx$$

on  $H_a^{\alpha/2}(0, T)$ , which we refer to as the *kinetic* and *potential* energies, respectively. For  $\alpha > 1$  the fNLS (1.1) admits the conserved quantities

$$\begin{aligned} \mathcal{H}(u) &:= K(u) + P(u) = \frac{1}{2} \int_0^T \left( |\Lambda^{\alpha/2} u|^2 + \frac{1}{\sigma + 1} |u|^{2\sigma+2} \right) dx, \\ Q(u) &:= \frac{1}{2} \int_0^T |u|^2 dx, \quad N(u) := \frac{i}{2} \int_0^T \overline{\Lambda^{1/2} u} H \Lambda^{1/2} u dx \end{aligned}$$

which we refer to as the *Hamiltonian (energy)*, *charge*, and *(angular) momentum*<sup>2</sup>, respectively. Conservation of  $\mathcal{H}$  comes from the fact that (1.1) is autonomous in time, while conservation of  $Q$  and  $N$  is due to the phase and translational invariance of (1.1), respectively. Above,  $H$  denotes the Hilbert transform, a bounded linear map from  $L_a^2(0, T) \rightarrow L_a^2(0, T)$  with unit norm.

For a general  $\alpha \in (1, 2]$ , it is clear that  $\mathcal{H}$ ,  $Q$ , and  $N$  are smooth functionals on  $H_a^{\alpha/2}(0, T)$ . Further, their first order variational derivatives are smooth maps from  $H_a^{\alpha/2}(0, T)$  into  $H_a^{\alpha/2}(0, T)^*$  given explicitly by

$$\delta H(u) = \Lambda^\alpha u + |u|^{2\sigma} u, \quad \delta Q(u) = u, \quad \delta N(u) = iu'.$$

It follows from (2.1) that  $T$ -antiperiodic standing waves of (1.1) arise as critical points of the Lagrangian functional

$$H_a^{\alpha/2}(0, T) \ni u \mapsto \mathcal{H}(u) + \omega Q(u) + cN(u)$$

for some  $\omega, c \in \mathbb{R}$ . It is natural now to treat the parameters  $\omega$  and  $c$  as Lagrange multipliers, and search for solutions of (2.1) as critical points of  $\mathcal{H}$  subject to the conservation of  $Q$  and  $N$ . However, below we need precise information on the range of values of  $c$  for which such a critical point exists. Consequently we find it more appropriate to treat the wave speed as a free-parameter and to attempt to construct solutions of (2.1) for a *fixed*  $c$  as critical points of the functional

$$F_c(u) := \mathcal{H}(u) + cN(u)$$

subject to fixed  $Q$ . The fact that such critical points exist as constrained minimizers is guaranteed by the following.

**Proposition 2.1.** *Let  $\alpha \in (1, 2)$  and  $T, \sigma > 0$  be fixed in the defocusing ( $\gamma = -1$ ) fNLS (1.1). For each  $\mu > 0$  define the constraint space*

$$\mathcal{A}_\mu := \left\{ u \in H_a^{\alpha/2}(0, T) : Q(u) = \mu \right\}.$$

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<sup>2</sup>On smooth solutions, the momentum functional is defined as  $N(u) = \frac{i}{2} \int_0^T \bar{u}(z) u'(z) dz$ . Using the identity  $\partial_x = H\Lambda$ , we may consider  $u_x$  to be well-defined on  $H_a^{\alpha/2}(0, T)$  in the sense of distributions.

Then for each  $\mu > 0$  and  $|c| < c_* := \left(\frac{\pi}{T}\right)^{\alpha-1}$  there exists a nontrivial  $\phi = \phi(\cdot; c, \mu) \in \mathcal{A}$  such that

$$F_c(\phi) = \min_{u \in \mathcal{A}_\mu} F_c(u),$$

and  $\phi(\cdot; c, \mu)$  satisfies (2.1) for some  $\omega = \omega(c, \mu) \in \mathbb{R}$  in the sense of distributions. The functions  $\phi$  and  $\omega$  depend on  $c$  and  $\mu$  in a  $C^1$  manner and  $\phi \in H_a^\infty(0, T)$ . Moreover,  $\phi$  minimizes the Lagrangian functional

$$(2.5) \quad \mathcal{E}(u; c, \mu) := \mathcal{H}(u) + \omega(c, \mu)Q(u) + cN(u)$$

over  $H_a^{\alpha/2}(0, T)$  subject to the fixed  $Q$  and  $N$ ; specifically

$$\mathcal{E}(\phi; c, \mu) = \inf \{ \mathcal{E}(\psi; c, \mu) : \psi \in \mathcal{A}_\mu, \quad N(\psi) = N(\phi) \}.$$

*Proof.* First, for  $\alpha > 1$  we observe Cauchy-Schwarz and the continuity of the embedding  $H_a^{\alpha/2}(0, T) \subset H_a^{1/2}(0, T)$  imply that  $|N(u)| \leq \left(\frac{T}{\pi}\right)^{\alpha-1} K(u)$  for all  $u \in H_a^{\alpha/2}(0, T)$ . It follows that the functional  $F_c$  is bounded below on  $\mathcal{A}$  for all  $|c| \leq \left(\frac{\pi}{T}\right)^{\alpha-1}$ :

$$F_c(u) = K(u) + P(u) + cN(u) \geq \left(1 - |c| \left(\frac{T}{\pi}\right)^{\alpha-1}\right) K(u).$$

Thus, if  $|c| \leq \left(\frac{\pi}{T}\right)^{\alpha-1}$  then the quantity  $\lambda := \inf_{u \in \mathcal{A}_\mu} F_c(u)$  is well defined and finite, hence there exists a minimizing sequence  $\{u_k\}_{k=1}^\infty \subset \mathcal{A}_\mu$  such that  $\lim_{k \rightarrow \infty} F_c(u_k) = \lambda$ . Furthermore, using that  $\|u\|_{H_a^{\alpha/2}(0, T)}^2 = 2K(u) + 2Q(u)$  we have

$$\frac{1}{2} \left(1 - \left(\frac{T}{\pi}\right)^{\alpha-1} |c|\right) \|u_k\|_{H_a^{\alpha/2}(0, T)}^2 \leq F_c(u_k) + \left(1 - \left(\frac{T}{\pi}\right)^{\alpha-1} |c|\right) \mu,$$

from which it follows that if  $|c| < \left(\frac{\pi}{T}\right)^{\alpha-1}$  then  $\{u_k\}_{k=1}^\infty$  is a bounded sequence in  $H_a^{\alpha/2}(0, T)$ . By Banach-Alaoglu and the fact that weak limits are unique, we may thus extract a subsequence  $\{u_{k_j}\}_{j=1}^\infty$  and a function  $\phi \in H_a^{\alpha/2}(0, T)$  such that  $u_{k_j}$  converges weakly to  $\phi$  in  $H_a^{\alpha/2}(0, T)$  and strongly (in norm) to  $\phi$  in  $L_a^2(0, T)$ . Since  $H_a^{\alpha/2}(0, T)$  is compactly embedded in  $L_a^{2(\sigma+1)}(0, T)$  for any  $\sigma > 0$  by Sobolev embedding, we have that  $P$  is a compact operator on  $H_a^{\alpha/2}(0, T)$ . Together with the fact that  $K$  is lower semicontinuous with respect to weak convergence in  $H_a^{\alpha/2}(0, T)$ , the above observations imply that

$$(2.6) \quad \liminf_{j \rightarrow \infty} \mathcal{H}(u_{k_j}) = \liminf_{j \rightarrow \infty} (K(u_{k_j}) + P(u_{k_j})) \geq \mathcal{H}(\phi), \quad Q(\phi) = \mu.$$

Finally, noting that  $H_a^{\alpha/2}(0, T)$  is compactly embedded into  $H_a^{1/2}(0, T)$  by Rellich-Kondrachov, the sequence  $\{u_{k_j}\}$  furthermore converges to  $\phi$  strongly in  $H_a^{1/2}(0, T)$ . From the estimate

$$|N(u) - N(v)| \leq \frac{1}{2} \left( \|u\|_{H_a^{1/2}(0, T)} + \|v\|_{H_a^{1/2}(0, T)} \right) \|u - v\|_{H_a^{1/2}(0, T)},$$

it follows that  $N(u_{k_j}) \rightarrow N(\phi)$  as  $j \rightarrow \infty$ . It now follows from (2.6) that  $\phi \in \mathcal{A}_\mu$  and

$$\lambda \leq F_c(\phi) \leq \liminf_{j \rightarrow \infty} F_c(u_{k_j}) = \lambda,$$



so that  $F_c(\phi) = \lambda$ . Clearly then  $\phi$  solves (2.1) for some  $\omega \in \mathbb{R}$ , and both  $\phi$  and  $\omega$  depend on  $c$  and  $\mu$  in a  $C^1$  manner.

To see that  $\phi(\cdot; c, \mu)$  minimizes  $\mathcal{E}(\cdot; \mu, c)$  over  $H_a^{\alpha/2}(0, T)$  constrained to  $Q(u) = \mu$  and  $N(u) = N(\phi)$ , simply observe that  $\mathcal{E} = F_c + \omega(c, \mu)Q$  so that  $\phi$  clearly minimizes  $\mathcal{E}(\cdot; c, \mu)$  over  $\mathcal{A}_\mu$ . In particular,  $\mathcal{E}(\phi; c, \mu) \leq \mathcal{E}(\psi; c, \mu)$  for all  $\psi \in \mathcal{A}_\mu$  with  $N(\psi) = N(\phi)$ .

It remains to establish the smoothness of the solution  $\phi$ . By construction, we know that  $\phi \in H_a^{\alpha/2}(0, T)$ . To show that  $\phi \in H_a^\alpha(0, T)$ , notice that for any  $|c| < (\frac{\pi}{T})^{\alpha-1}$  the profile equation can be written as

$$-\phi = (\Lambda^\alpha + ic\partial_x)^{-1} (\omega\phi + |\phi|^{2\sigma}\phi),$$

where the operator  $(\Lambda^\alpha + ic\partial_x)^{-1}$  is well-defined since  $\phi$  has zero mean by antiperiodicity. Since  $H_a^{\alpha/2}(0, T) \subset L^\infty(0, T)$  by Sobolev embedding, the Plancherel theorem yields

$$\begin{aligned} \|\Lambda^\alpha \phi\|_{L^2(0, T)} &= \left\| \frac{|n|^\alpha}{|n|^\alpha - (\frac{T}{\pi})^{\alpha-1} cn} \left( \omega \hat{\phi}(n) + \widehat{|\phi|^{2\sigma}\phi}(n) \right) \right\|_{\ell^2(\mathbb{Z} \setminus \{0\})} \\ &\leq C (\|\phi\|_{L^2(0, T)} + \|\phi\|_{L^2(0, T)}^{2\sigma}) \\ &\leq C \left( \|\phi\|_{L^2(0, T)} + \|\phi\|_{L^\infty(0, T)}^{2\sigma} \|\phi\|_{L^2(0, T)} \right) < \infty, \end{aligned}$$

for some  $C > 0$  independent of  $\phi$ , and hence  $\phi \in H_a^\alpha(0, T)$ . Similarly, by the fractional chain rule [30, (3.3)] we have

$$\begin{aligned} \|\Lambda^{2\alpha} \phi\|_{L^2(0, T)} &\leq C \left( \|\Lambda^\alpha \phi\|_{L^2(0, T)} + \|\Lambda^\alpha (|\phi|^{2\sigma}\phi)\|_{L^2(0, T)} \right) \\ &\leq C (\|\Lambda^\alpha \phi\|_{L^2(0, T)} + \|\phi\|_{L^\infty(0, T)}^{2\sigma} \|\Lambda^\alpha \phi\|_{L^2(0, T)}) < \infty \end{aligned}$$

for some  $C > 0$  independent of  $\phi$ , so that  $\phi \in H_a^{2\alpha}(0, T)$ . Iterating, we find that  $\phi \in H_a^\infty(0, T)$  as claimed.  $\square$

To recapitulate, for each  $\alpha \in (1, 2)$ ,  $\sigma, T > 0$ ,  $|c| < c_*$  and  $\mu > 0$ , Proposition 2.1 produces a generally complex-valued function  $\phi(\cdot; c, \mu) \in H_a^\infty(0, T)$  and a  $\omega(c, \mu) \in \mathbb{R}$  such that

$$\Lambda^\alpha \phi + \omega(c, \mu)\phi + ic\phi' + |\phi|^{2\sigma}\phi = 0, \quad Q(\phi) = \mu.$$

In particular, incorporating phase and translation invariance, for each half-period  $T > 0$  we have constructed a four-parameter family of generally complex-valued  $T$ -antiperiodic smooth solutions of the defocusing fNLS (1.1):

$$u(x, t; c, \mu, \theta, \zeta) = e^{i(\omega(c, \mu)t - \theta)} \phi(x - ct + \zeta; c, \mu)$$

where  $|c| < c_*$ ,  $\mu > 0$ ,  $\theta \in [0, 2\pi]$  and  $\zeta \in \mathbb{R}$ . These solutions are parameterized by the wave speed  $c$ , the charge  $Q(u)$  of the wave, and the parameters  $\theta$  and  $\zeta$  associated to the continuous Lie point symmetries of the governing nonlocal PDE. As stated at the beginning of this section, our focus in the remainder of the paper is on the *standing* wave solutions of (1.1), corresponding to the above solutions with  $c = 0$ . Properties of the functions  $\phi$  and  $\omega$  at  $c = 0$  are recorded in the following lemma.

**Lemma 2.2.** *The function  $\omega : (-c_*, c_*) \times \mathbb{R}_+ \rightarrow \mathbb{R}$  constructed in Proposition 2.1 is even in  $c$  and, for each  $\mu > 0$ , the profile  $\phi(\cdot; 0, \mu) \in H_a^\infty(0, T)$  can be taken to be real-valued, even, and strictly decreasing on  $(0, T)$  and they satisfy the constrained variational problem*

$$\mathcal{E}(\phi; 0, \mu) = \inf \left\{ \mathcal{E}(\psi; 0, \mu) : \psi \in H_a^{\alpha/2}(0, T), \quad Q(\psi) = Q(\phi), \quad N(\psi) = 0 \right\}.$$

Furthermore, for each  $\mu > 0$  the function  $\frac{\partial \phi}{\partial c}(\cdot; 0, \mu)$  is purely imaginary.

*Remark 2.3.* In the classical case  $\alpha = 2$ , the evenness of  $\omega(c, \mu)$  in  $c$  and the realness of the function  $i \frac{\partial \phi}{\partial c}$  follow trivially from the Galilean invariance (2.2).

*Proof.* Observe that for all  $\psi \in H_a^{\alpha/2}(0, T)$  we have  $F_{-c}(\psi) = F_c(\bar{\psi})$ . Thus, if  $\phi$  minimizes  $F_c$  with fixed  $Q$ , then  $\bar{\phi}$  minimizes  $F_{-c}$  with the same fixed  $Q$ : in particular,  $\phi(\cdot; -c, \mu) = \bar{\phi}(\cdot; c, \mu)$ . Thus,  $\phi$  and  $\bar{\phi}$  solve the associated Euler-Lagrange equations

$$\Lambda^\alpha \phi + \omega(c, \mu) \phi + ic\phi' + |\phi|^{2\sigma} \phi = 0$$

and

$$\Lambda^\alpha \bar{\phi} + \omega(-c, \mu) \bar{\phi} - ic\bar{\phi}' + |\bar{\phi}|^{2\sigma} \bar{\phi} = 0.$$

Multiplying the first equation by  $\bar{\phi}$ , the second by  $\phi$ , integrating, and subtracting we find

$$(\omega(c, \mu) - \omega(-c, \mu)) \int_0^T |\phi|^2 dx = 0$$

from which it follows that  $\omega$  is an even function of  $c$ , as claimed. Similarly, we find

$$\operatorname{Re}(\phi(\cdot; c, \mu)) = \frac{1}{2} (\phi(\cdot; c, \mu) + \phi(\cdot; -c, \mu)),$$

which is an even function of  $c$ . Differentiating this relation with respect to  $c$  at  $c = 0$  yields  $\operatorname{Re}\left(\frac{\partial \phi}{\partial c}(\cdot; 0, \mu)\right) = 0$ .

Finally, the fact that  $\phi(\cdot; 0, \mu)$  can be taken to be real-valued and non-negative on  $(-T/2, T/2)$  follows from the Pólya-Szegő inequality established in Appendix A, which states that

$$F_0(\psi) \geq F_0(|\psi|), \quad Q(\psi) = Q(|\psi|), \quad \forall \psi \in H_a^{\alpha/2}(0, T)$$

The monotonicity of  $\phi(\cdot; 0, \mu)$  then follows from  $2T$ -periodic rearrangement arguments outlined in Appendix A, and the fact that  $\phi(\cdot; 0, \mu)$  satisfies the stated constrained minimization problem follows trivially from Proposition 2.1 by observing that  $N(f) = 0$  for all real-valued  $f \in H_a^{\alpha/2}(0, T)$ .  $\square$

In conclusion, for fixed  $\alpha \in (1, 2)$  and  $\sigma > 0$ , we have constructed for each  $T > 0$  a three-parameter family of *real-valued,  $T$ -antiperiodic, even* solutions of the profile equation (2.1) with  $c = 0$ . These profiles lead to a three-parameter family of *standing wave* solutions of (1.1) of the form

$$u(x, t; \mu, \theta, \zeta) = e^{i(\omega(0, \mu)t + \theta)} \phi(x + \zeta; 0, \mu)$$

In the forthcoming analysis, we will restrict our attention to these real-valued profiles with  $c = 0$ . However, as stated previously, the fact that such solutions belong to a larger class of complex-valued traveling waves will be used heavily in the forthcoming analysis; see Remark 3.13. For notational simplicity, in the sequel we will suppress the dependence of  $\omega$  and  $\phi$  on the wave speed  $c$  whenever it is clear from context that  $c = 0$ .

### 3 Nondegeneracy of the Linearization in the Defocusing Case

Throughout this section, for each  $\mu > 0$  we let  $\phi(x; \mu)$  denote a real-valued, even,  $T$ -antiperiodic standing wave solution of the nonlocal profile equation (2.1) with  $c = 0$  satisfying  $Q(\phi) = \mu$ , whose existence is guaranteed by Proposition 2.1 and Lemma 2.2, so that the function  $u(x, t; \mu) = e^{i\omega(\mu)t} \phi(x; \mu)$  is a  $T$ -antiperiodic standing wave solution of the defocusing ( $\gamma = -1$ ) fNLS (1.1). Moving to a co-rotating coordinate frame, the profile  $\phi(\cdot; \mu)$  is thus a real-valued,  $T$ -antiperiodic equilibrium solution of the PDE

$$(3.1) \quad iu_t - \omega(\mu)u - \Lambda^\alpha u + \gamma|u|^{2\sigma}u = 0,$$

which can be rewritten as the Hamiltonian system

$$u_t = -i\delta\mathcal{E}(u; c, \mu)$$

acting on  $L_{\text{per}}^2(0, 2T)$ , where here  $\mathcal{E}$  is the modified energy functional defined in (2.5). For such Hamiltonian systems, it is well known that the local dynamics of (3.1) near  $\phi$ , in particular its orbital stability or instability, is intimately related to spectral properties of the second variation of the energy functional

$$(3.2) \quad \delta^2\mathcal{E}(\phi; c, \mu) = \Lambda^\alpha + \omega(\mu) + ic\partial_x - \gamma|\phi|^{2\sigma} - 2\gamma\sigma|\phi|^{2\sigma-2}\text{Re}(\bar{\phi}\cdot)$$

acting on appropriate subspaces of  $L_{\text{per}}^2(0, 2T)$ . Of particular importance, observe that the  $T$ -antiperiodicity of  $\phi$  implies that the operator  $\delta^2\mathcal{E}(\phi)$  has  $T$ -periodic coefficients. As we will see below, however, the continuous Lie point symmetries of (3.1) generate elements of the kernel of  $\delta^2\mathcal{E}(\phi)$  that are  $T$ -antiperiodic, and hence zero is an isolated eigenvalue of  $\delta^2\mathcal{E}(\phi)$  acting on  $L_a^2(0, T)$  with finite multiplicity. In the forthcoming analysis, we restrict our attention to  $T$ -antiperiodic perturbations of the underlying wave  $\phi$ , thus requiring a detailed spectral analysis of the operator  $\delta^2\mathcal{E}(\phi)$  acting on  $L_a^2(0, T)$ .

To aid in our analysis, we find it convenient to decompose the action of  $\delta^2\mathcal{E}(\phi)$  into real and imaginary parts. Restricting our attention to the real-valued, stationary ( $c = 0$ ) solutions  $\phi$  constructed in Lemma 2.2, the operator  $\delta^2\mathcal{E}(\phi)$  acts as a diagonal operator on  $L_a^2(0, T)$ . Indeed, for a given  $v \in H_a^\alpha(0, T)$ , decomposing  $v = a + bi$  for  $a, b$  real-valued we can write

$$\delta^2\mathcal{E}(\phi)v = L_+a + iL_-b$$

where the operators  $L_\pm$  are linear operators acting on  $L_a^2([0, T]; \mathbb{R})$  defined by

$$(3.3) \quad L_+ := \Lambda^\alpha - \gamma(2\sigma + 1)\phi^{2\sigma} + \omega$$

$$(3.4) \quad L_- := \Lambda^\alpha - \gamma\phi^{2\sigma} + \omega.$$

Consequently, we can consider  $\delta^2\mathcal{E}(\phi)$  as the matrix operator  $\text{diag}(L_+, L_-)$  acting on the product space  $L_a^2([0, T]; \mathbb{R})^2$ . Concerning the spectrum of  $\delta^2\mathcal{E}(\phi)$ , observe that  $\delta^2\mathcal{E}(\phi)$  is bounded below and self-adjoint on  $L_a^2(0, T)$  with compactly embedding domain  $H_a^\alpha(0, T)$ . Consequently, the spectrum of  $\delta^2\mathcal{E}(\phi)$ , and hence of the operators  $L_\pm$ , acting on  $L_a^2(0, T)$  is comprised of a countably infinite discrete set of real eigenvalues tending to  $+\infty$  with no finite accumulation point. Key information in the stability analysis of  $\phi$  now rests on determining the number of negative  $T$ -antiperiodic eigenvalues of  $\delta^2\mathcal{E}(\phi)$ , referred to as the *Morse index* of the operator  $\delta^2\mathcal{E}(\phi)$ , as well as a characterization of its  $T$ -antiperiodic kernel.

Note that, by the translation and phase invariance of (3.1), it is straightforward to verify that

$$L_+\phi' = 0 \quad \text{and} \quad L_-\phi = 0,$$

and hence  $\phi'$  and  $\phi$  belong to the  $T$ -antiperiodic kernel of the  $T$ -periodic coefficient operators  $L_+$  and  $L_-$ , respectively. In general, it is very difficult to determine whether or not these functions comprise the entirety of the  $T$ -antiperiodic kernels. Indeed, it is not difficult to construct examples in the local case  $\alpha = 2$  where either  $L_\pm$  have kernels with higher multiplicity. In the antiperiodic case, the issue is even further complicated by the fact that standard Perron-Frobenius arguments fail to characterize the ground state eigenvalues of  $\delta^2\mathcal{E}(\phi)$  on  $L_a^2(0, T)$ . Indeed, even in the local case  $\alpha = 2$  it is not difficult to construct examples where the first antiperiodic eigenvalue of a Schrödinger operator with periodic potential has algebraic multiplicity two. Furthermore, determining the number of  $T$ -antiperiodic negative eigenvalues of  $\delta^2\mathcal{E}(\phi)$  is often handled by classical Sturm-Liouville type arguments. Indeed, in the classical case  $\alpha = 2$  the fact that both  $\phi$  and  $\phi'$  have roots in  $[0, T)$  implies that  $\lambda = 0$  may be either the first or second  $T$ -antiperiodic eigenvalue of  $L_\pm$ , and hence a-priori the operator  $\delta^2\mathcal{E}(\phi)$  may have at most *two* negative  $T$ -antiperiodic eigenvalues, which is typically an unfavorable energy configuration for orbital stability. When  $\alpha \in (1, 2)$ , classical Sturm-Liouville arguments do not apply to the operators  $L_\pm$  and hence new methods will be necessary to determine the number of negative eigenvalues of  $\delta^2\mathcal{E}(\phi)$ . Nevertheless, the main result for this section is the following.

**Proposition 3.1** (Nondegeneracy & Morse Index Bounds). *Let  $\alpha \in (1, 2)$  and  $\sigma > 0$  in the defocusing ( $\gamma = -1$ ) fNLS (1.1). If  $\phi(\cdot; \mu) \in H_a^{\alpha/2}(0, T)$  is a real-valued local minimizer of  $\mathcal{H}$  over  $H_a^{\alpha/2}(0, T)$  subject to fixed  $Q(u) = \mu > 0$  and  $N(u) = 0$ , then the associated Hessian operator  $\delta^2\mathcal{E}(\phi)$  acting on  $L_a^2(0, T)$  is nondegenerate, i.e.*

$$\ker(\delta^2\mathcal{E}(\phi)) = \text{span}\{\phi', i\phi\}$$

and<sup>3</sup>  $n_-(\delta^2\mathcal{E}(\phi)) = 1$ . Specifically, the operators  $L_\pm$  are nondegenerate acting on  $L_a^2(0, T)$  with

$$\ker(L_+) = \text{span}\{\phi'\} \quad \text{and} \quad \ker(L_-) = \text{span}\{\phi\}$$

and, further, we have  $n_-(L_+) = 0$  and  $n_-(L_-) = 1$ .

As noted in the introduction, Proposition 3.1 was established using ODE techniques in the local case  $\alpha = 2$  by Gallay and Haragus [20]. Precisely, their proof utilizes Sturm-Liouville theory for (local) differential operators, together with a homotopy argument and a-priori control over the dimension of the  $T$ -antiperiodic kernels. While these ODE-based techniques are not directly available in the nonlocal setting  $\alpha \in (0, 2)$ , we recall that Frank and Lenzmann [18] recently obtained the nondegeneracy of the linearization about solitary waves for a family of nonlinear nonlocal models that include the *focusing* ( $\gamma = +1$ ) fNLS (1.1). Their idea was to find a suitable substitute for the Sturm-Liouville oscillation theory to control the number of sign changes in eigenfunctions for a fractional Schrödinger operator with real, localized potential. This theory, developed on the line, was then adapted to the periodic setting in [42], where the authors considered the nonlinear orbital stability of  $T$ -periodic traveling wave solutions to the fractional KdV equation.

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<sup>3</sup>Here and throughout, for a given linear operator  $\mathcal{L}$  on  $L_a^2(0, T)$  we denote the Morse index of  $\mathcal{L}$  as  $n_-(\mathcal{L}) := \#\{\lambda \in \sigma_{L_a^2(0, T)}(\mathcal{L}) : \lambda < 0\}$ .

The proof of Proposition 3.1 extends these previous nondegeneracy results to encompass the  $T$ -antiperiodic spectra of fractional Schrödinger operators with real,  $T$ -periodic potentials. As mentioned above, this extension is significant as, even in the classical  $\alpha = 2$  case, the ground state antiperiodic eigenvalue of  $T$ -periodic linear Schrödinger type operators need not be simple. This is in stark contrast to the ground state  $T$ -periodic eigenvalues of such operators, which are *always simple* by Perron-Frobenius theory. While our proof follows the basic strategy in [18] and [42], substantial modifications are necessary to accommodate the antiperiodic structure of the admissible class of perturbations.

There are two key analytical results necessary to establish Proposition 3.1. First, we require an appropriate characterization of the ground state eigenfunctions of  $L_{\pm}$  acting on  $L^2_a(0, T)$ . A natural approach is to attempt to use a Perron-Frobenius type argument, demonstrating that the semigroups  $e^{-L_{\pm}t}$  are positivity improving on appropriate subspaces of  $L^2_a(0, T)$ . Second, we require a nonlocal Sturm-Liouville type oscillation theory for the second antiperiodic eigenfunctions of  $L_{\pm}$ . Following the general ideas in [18] and [42], this is accomplished by extending the antiperiodic eigenvalue problems for  $L_{\pm}$  on  $L^2_a(0, T)$  to appropriate local problems on the upper half-space.

### 3.1 Perron-Frobenius Theory for Antiperiodic Eigenfunctions

The goal of this section is to provide a characterization of the antiperiodic ground state eigenfunctions for linear, fractional Schrödinger operators of the form

$$(3.5) \quad L := \Lambda^{\alpha} + V(x),$$

where the potential  $V(x)$  is even, real-valued, smooth and  $T$ -periodic for some finite  $T > 0$ . In particular, we will classify properties of the  $T$ -antiperiodic ground state for  $L$ , along with upper bounds on the number sign changes on higher  $T$ -antiperiodic eigenfunctions. As noted in the introduction, even in the local case  $\alpha = 2$  such results are nontrivial as  $T$ -antiperiodic ground states need not be simple. As we will see below, this comes from the fact that the semigroup generated by  $L$  is not positivity improving when acting on  $L^2_a(0, T)$ . To handle this difficulty, we will decompose the space  $L^2_a(0, T)$  of real-valued  $T$ -antiperiodic functions on  $\mathbb{R}$  into the (invariant) even and odd subspaces, and develop ground state and oscillation theories for the operator  $L$  in each sector separately. As we will see, restricted to these sectors, the semigroup generated by  $L$  will indeed be positivity improving. Finally, using rearrangement properties we find an ordering between the antiperiodic odd and even ground state eigenvalues for  $L$  in terms of monotonicity properties of the potential  $V$  on  $(0, T)$ .

We begin by observing that the  $T$ -periodicity of the potential  $V$  implies that the operator  $L$  is well defined as a closed, densely defined operator from  $L^2_a(0, T)$  into itself. Since  $V$  is a bounded and smooth potential, the operator  $L$  is a relatively compact perturbation of the operator  $-\Lambda^{\alpha}$ , Theorem XIII.44 from [46] implies that the ground state eigenvalues of  $L$  acting on an invariant subspace  $\mathcal{Y}$  of  $L^2_a(0, T)$  is simple as an eigenvalue of  $L|_{\mathcal{Y}}$  provided the fractional heat semigroup  $\{e^{-\Lambda^{\alpha}t}\}_{t \geq 0}$  is positivity improving on  $\mathcal{Y}$ ; that is, if

$$f \in \mathcal{Y}, \quad f \geq 0, \quad f \neq 0 \quad \implies \quad e^{-\Lambda^{\alpha}t}f > 0 \quad \text{on } \mathcal{Y}.$$

Thus, it is sufficient to study the semigroup generated by  $-\Lambda^{\alpha}$  on  $L^2_a(0, T)$ , which we shall study below by first considering the semigroup acting on  $L^2(\mathbb{R})$  and then considering appropriate periodizations of its integral kernel.

The semigroup  $e^{-\Lambda^\alpha t}$  acting on  $L^2(\mathbb{R})$  is naturally understood via the Fourier transform. Throughout, the operator  $\mathcal{F}$  will denote the extension to the space of tempered distributions  $\mathcal{S}'(\mathbb{R})$  of the Fourier transform

$$\mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx$$

on the Schwartz space  $\mathcal{S}(\mathbb{R})$ , with inverse  $\mathcal{F}^{-1}(f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} f(\xi) d\xi$ . In particular, with this particular normalization  $\mathcal{F}$  defines a unitary operator on  $L^2(\mathbb{R})$ . For all  $t \geq 0$ , the operators  $e^{-\Lambda^\alpha t}$  acting on  $\mathcal{S}(\mathbb{R})$  can be understood via

$$(3.6) \quad e^{-\Lambda^\alpha t} f(x) = \mathcal{F}^{-1} \left( e^{-|\cdot|^\alpha t} \widehat{f}(\cdot) \right) (x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-|\xi|^\alpha t} \widehat{f}(\xi) e^{i\xi x} d\xi.$$

Alternatively, introducing the function

$$K(x, t) := \mathcal{F}^{-1} \left( e^{-|\cdot|^\alpha t} \right) (x)$$

we see that  $e^{-\Lambda^\alpha t}$  can be viewed as the convolution operator

$$e^{-\Lambda^\alpha t} f(x) = \int_{\mathbb{R}} K(x - y, t) f(y) dy.$$

Note that, when  $\alpha = 2$ , the integral kernel  $K$  agrees with the standard heat-kernel and can be explicitly expressed in terms of a Gaussian function. While such explicit formulas are not available in the nonlocal case  $\alpha \in (0, 2)$ , in the recent work of Frank & Lenzmann [18, Appendix A] it was observed that, for all  $t > 0$  and  $\alpha \in (0, 2)$ , the kernel  $K(\cdot, t)$  is even and strictly positive with  $\partial_x K(x, t) < 0$  for all  $x > 0$  and, furthermore, decays rapidly at spatial infinity. Further, we know that  $K(\cdot, t) \in L^1(\mathbb{R})$  since, by the positivity of  $K$ ,

$$\|K(\cdot, t)\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} K(x, t) dx = \mathcal{F}(K(\cdot, t))(\xi = 0) = 1.$$

Since our interest is in a  $T$ -antiperiodic oscillation theory for operators of the form (3.5) we now describe how  $e^{-\Lambda^\alpha t}$  acts on periodic functions. Since  $K(\cdot, t)$  lies in  $L^1(\mathbb{R})$  for all  $t > 0$ , given any  $f \in L^\infty(\mathbb{R})$  that is  $2T$ -periodic we can write

$$e^{-\Lambda^\alpha t} f(x) = \int_{-T}^T \left( \sum_{n \in \mathbb{Z}} K(x - y + 2nT, t) \right) f(y) dy = \int_{-T}^T K_p(x - y, t) f(y) dy,$$

where

$$(3.7) \quad K_p(x, t) := \sum_{n \in \mathbb{Z}} K(x + 2nT, t)$$

represents the  $2T$ -periodic *periodization* of the integral kernel  $K$ . Observe that the sum defining  $K_p$  is absolutely convergent for each  $t > 0$  due to the rapid decay of  $K(\cdot, t)$  at spatial infinity. Furthermore, one can show that  $K_p \in L^1_{\text{per}}(0, 2T)$  and admits the Fourier series expansion

$$K_p(x, t) = \sum_{n \in \mathbb{Z}} e^{-|\pi n/T|^\alpha t} e^{\pi i n x / T}$$

so that, by the convolution theorem, we have that  $e^{-\Lambda^\alpha t}$  acts on  $2T$ -periodic functions via

$$(3.8) \quad e^{-\Lambda^\alpha t} f(x) = \sum_{n \in \mathbb{Z}} e^{-|\pi n/T|^\alpha t} \hat{f}(n) e^{\pi i n x/T}$$

In particular, the operator  $e^{-\Lambda^\alpha t}$  on  $2T$ -periodic functions has the exact same Fourier symbol as when considering the same operator on  $L^2(\mathbb{R})$ .

While the Fourier representation (3.8) seems useful for numerical calculations, we are unfortunately unable to extract from it the necessary information for our forthcoming theory. Nevertheless, using a slightly different representation coming from Bernstein's theorem, we have the following.

**Lemma 3.2.** *For all  $t > 0$  and  $\alpha \in (0, 2]$ ,  $K_p(\cdot, t)$  is positive, even,  $2T$ -periodic, and strictly decreasing on  $[0, T]$ .*

*Remark 3.3.* It was recently shown in [17] that the periodization of a function  $f \in L^1(\mathbb{R})$  that is even and completely monotone<sup>4</sup> on  $(0, \infty)$  is automatically even and completely monotone on a half period. However, as is evident from the subordination formula (3.9) below, the kernel  $K(x)$  is *not* completely monotone on  $(0, \infty)$ , and hence this abstract result does not apply.

*Proof.* The whole line kernel  $K(x, t)$  was shown by Frank & Lenzmann [18, Appendix A] to be even and positive for all  $t > 0$ ,  $x \in \mathbb{R}$ . Since  $K(\cdot, t) \in L^1(\mathbb{R})$  for all  $t > 0$ , the representation (3.7) implies that the periodization  $K_p(\cdot, t)$  must also be even, positive and  $2T$ -periodic for all  $t > 0$ . To prove that  $K_p(\cdot, t)$  is decreasing on  $(0, T)$  for each  $t > 0$ , we follow [18] and observe that the function  $g(z) = e^{-z^{\alpha/2}}$  is completely monotone on the positive half-line  $(0, \infty)$  for all  $0 < \alpha \leq 2$  and hence, by Bernstein's theorem, is the Laplace transform of a non-negative finite measure  $\nu_\alpha$  depending on  $\alpha$ , i.e.  $e^{-z^{\alpha/2}} = \int_0^\infty e^{-\tau z} d\nu_\alpha(\tau)$  for some such measure  $\nu_\alpha$ . Setting  $z = |x|^2$  and recalling the inverse Fourier representation for the Gaussian  $e^{-\tau \xi^2}$  leads to<sup>5</sup> the “subordination formula”

$$(3.9) \quad K(x, t) = t^{-1/\alpha} \int_0^\infty \frac{1}{\sqrt{2\tau}} \exp\left(-\frac{t^{-2/\alpha} x^2}{4\tau}\right) d\nu_\alpha(\tau)$$

valid for all  $x \in \mathbb{R}$  and  $t > 0$ . From (3.7) it follows that for all  $\alpha \in (0, 2)$  the  $2T$ -periodic kernel  $K_p$  can be expressed as

$$\begin{aligned} K_p(x, t) &= t^{-1/\alpha} \int_0^\infty \frac{1}{\sqrt{2\tau}} \left[ \sum_{n \in \mathbb{Z}} \exp\left(-\frac{(x + 2nT)^2}{4t^{2/\alpha}\tau}\right) \right] d\nu_\alpha(\tau) \\ &= t^{-2/\alpha} \sqrt{2\pi} \int_0^\infty \left[ \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{(x + 2nT)^2}{4u}\right) \right] d\nu_\alpha(u), \end{aligned}$$

where the final equality follows from the variable substitution  $u = t^{2/\alpha}\tau$ . The integrand above may be recognized as the  $2T$ -periodized Gauss-Weierstrass kernel

$$(3.10) \quad \vartheta_u(x) := \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{(x + 2nT)^2}{4u}\right),$$

<sup>4</sup>That is,  $(-1)^j \partial_z^j g(z) \leq 0$  for all  $j \in \mathbb{N}$  and  $z > 0$ .

<sup>5</sup>The subordination formula in [18] is stated only for the case  $t = 1$ . This more general formula follows from the scaling  $K(x, t) = t^{-1/\alpha} K(t^{-1/\alpha} x, 1)$ .

hence we may express  $K_p(x, t)$  compactly as

$$K_p(x, t) = t^{-2/\alpha} \sqrt{2\pi} \int_0^\infty \vartheta_u(x) d\nu_\alpha(u).$$

The monotonicity properties of the function  $\vartheta_u(x)$  have been studied in [1, Theorem 4.2], where it was shown<sup>6</sup> to be strictly decreasing in  $x$  on  $(0, T)$  for all  $u > 0$ . It now clearly follows that if  $x, y \in (0, T)$  with  $x < y$ , then

$$K_p(x, t) - K_p(y, t) = t^{-2/\alpha} \sqrt{2\pi} \int_0^\infty (\vartheta_u(x) - \vartheta_u(y)) d\nu_\alpha(u) > 0,$$

i.e.  $K_p(\cdot, t)$  is decreasing on  $(0, T)$  for all  $t > 0$ , as claimed.  $\square$

*Remark 3.4.* A consequence of Bernstein's theorem for completely-monotone functions, the subordination formula (3.9) conveniently encodes all  $\alpha$ -dependence into the non-negative measure  $\nu_\alpha$ , which facilitates studying the fractional heat kernel using familiar techniques of the classical heat kernel. In order for this result to apply, we must restrict to  $\alpha \in (0, 2]$  as the function  $g(z) = \exp(-z^{\alpha/2})$  fails to be completely monotone for  $\alpha > 2$ .

To study the antiperiodic eigenvalues of  $L$ , we now further restrict the semigroup  $e^{-\Lambda^\alpha t}$  to the subspace  $L_a^2(0, T)$  of  $T$ -antiperiodic functions. For such  $f \in L_a^2(0, T)$  we have the representation

$$e^{-\Lambda^\alpha t} f(x) = \int_0^T [K_p(x - y, t) - K_p(x - y - T, t)] f(y) dy =: \int_0^T K_a(x - y, t) f(y) dy.$$

for the action of  $e^{-\Lambda^\alpha t}$  on  $T$ -antiperiodic functions, where  $K_a$  denotes the  $T$ -antiperiodic kernel

$$(3.11) \quad K_a(x, t) := K_p(x, t) - K_p(x - T, t),$$

Next, we gather some important properties of  $K_a$ .

**Lemma 3.5** (Properties of  $K_a$ ). *For all  $t > 0$  and  $\alpha \in (0, 2]$ , the function  $K_a(\cdot, t)$  is even,  $T$ -antiperiodic and strictly positive for all  $x \in (-T/2, T/2)$ . Furthermore,  $K_a(\cdot, t)$  is odd about  $x = T/2$  and is strictly decreasing on  $(0, T)$ .*

*Proof.* The parity and antiperiodicity of  $K_a$  follow directly from (3.11). Since all even,  $T$ -antiperiodic functions are odd<sup>7</sup> about  $x = T/2$ , it remains to show that  $K_a(\cdot, t)$  is positive on  $(-T/2, T/2)$  and strictly decreasing on  $(0, T)$ . To this end, fix  $x \in (0, T/2)$  and observe that the evenness of  $K_p(\cdot, t)$  implies that

$$(3.12) \quad K_a(x, t) = K_p(x, t) - K_p(T - x, t) > 0,$$

where the strict inequality follows since  $K_p(\cdot, t)$  is strictly decreasing on  $(0, T)$  by Lemma 3.2 and  $0 < x < T - x < T$  for  $x \in (0, T/2)$ . Since  $K_a(\cdot, t)$  is even for all  $t > 0$ , the positivity of  $K_a(\cdot, t)$  on  $(-T/2, T/2)$  follows. Similarly, differentiating (3.12) with respect to  $x$ , it follows that for  $x \in (0, T)$  we have

$$\partial_x K_a(x, t) = \partial_x K_p(x, t) + \partial_x K_p(T - x, t) < 0$$

where we have used that  $\partial_x K_p(x, t) < 0$  for all  $x \in (0, T)$  by Lemma 3.2.  $\square$

<sup>6</sup>While the results in [1] were stated only for the case  $T = \pi$  they easily extend to this more general setting via scaling.

<sup>7</sup>Indeed, if  $f$  is even and  $T$ -antiperiodic then  $f(x + T/2) = f(-x - T/2) = f(-x + T/2)$ .



An important consequence of Lemma 3.5 is that the semigroup  $e^{-\Lambda^\alpha t}$  is not positivity improving (nor even positivity preserving) on  $L_a^2(0, T)$ , and hence, by the functional calculus, the antiperiodic ground states of the operator  $L$  in (3.5) cannot be characterized by standard Perron-Frobenius arguments. However, we note that since the potential  $V(x)$  in (3.5) is even, the operator  $L$  respects the orthogonal decomposition

$$L_a^2(0, T) = L_{a,\text{even}}^2(0, T) \oplus L_{a,\text{odd}}^2(0, T),$$

where  $L_{a,\text{even/odd}}^2(0, T)$  denotes the sectors of even/odd functions in  $L_a^2(0, T)$ , respectively. Precisely, the sectors  $L_{a,\text{even/odd}}^2(0, T)$  are invariant subspaces for  $L$  and the above decomposition implies that all eigenfunctions for  $L$  may be chosen to be either even or odd. In particular,

$$\sigma_{L_a^2(0, T)}(L) = \sigma_{L_{a,\text{even}}^2(0, T)}(L) \cup \sigma_{L_{a,\text{odd}}^2(0, T)}(L),$$

where we emphasize the above spectral decomposition need not be disjoint. Next, we consider the action of the semigroup  $e^{-\Lambda^\alpha t}$  on the above invariant sectors.

First, note that if  $f \in L_{a,\text{even}}^2(0, T)$  then

$$\begin{aligned} e^{-\Lambda^\alpha t} f(x) &= \frac{1}{2} \left[ \int_0^T K_a(x-y, t) f(y) dy + \int_{-T}^0 K_a(x+y, t) f(y) dy \right] \\ &= \frac{1}{2} \left[ \int_0^T K_a(x-y, t) f(y) dy + \int_0^T K_a(x+y-T, t) f(y-T) dy \right] \\ &= \frac{1}{2} \int_0^T [K_a(x-y, t) + K_a(x+y, t)] f(y) dy \end{aligned}$$

where the final equality follows from the  $T$ -antiperiodicity of both  $K_a(\cdot, t)$  and  $f$ . Observe that since  $f(y)$  and  $K_a(x-y, t) + K_a(x+y, t)$  are even and  $T$ -antiperiodic in  $y$ , they are both odd functions in  $y$  about  $y = T/2$ . Consequently, their product is even in  $y$  about  $y = T/2$ , which yields the representation

$$(3.13) \quad e^{-\Lambda^\alpha t} f(x) = \int_0^{T/2} [K_a(x-y, t) + K_a(x+y, t)] f(y) dy.$$

for the action of semigroup  $e^{-\Lambda^\alpha t}$  on  $L_{a,\text{even}}^2(0, T)$ .

**Lemma 3.6.** *For all  $x, y \in (-T/2, T/2)$  and  $t > 0$ , we have*

$$K_a(x-y, t) + K_a(x+y, t) > 0.$$

*In particular, the semigroup  $e^{-\Lambda^\alpha t}$  restricted to  $L_{a,\text{even}}^2(0, T)$  is positivity improving, i.e. if  $f \in L_{a,\text{even}}^2(0, T)$  is non-trivial with  $f(x) \geq 0$  for  $x \in (-T/2, T/2)$ , then  $e^{-\Lambda^\alpha t} f(x) > 0$  for all  $x \in (-T/2, T/2)$ .*

*Proof.* We begin by proving the claim for  $x, y \in (0, T/2)$ . Fix  $t > 0$  and define  $G(x, y; t) := K_a(x-y, t) + K_a(x+y, t)$ , and note that  $G(x, y; t) = G(y, x; t)$  for all  $x, y$ . So, without loss of generality we need only prove that  $G(x, y; t) > 0$  for all  $(x, y) \in \mathcal{R} := \{(x, y) : 0 < x < T/2, 0 < y \leq x\}$ . Observe that for all  $(x, y) \in \mathcal{R}$ , we have

$$0 \leq x-y < T/2 \quad \text{and} \quad 0 \leq x+y < T,$$

hence

$$\partial_x G(x, y; t) = \partial_x K_a(x + y, t) + \partial_x K_a(x - y, t) < 0$$

since  $K_a(\cdot, t)$  is decreasing on  $(0, T)$  by Lemma 3.5. Moreover, for all  $y \in (0, T/2)$ , we have

$$G(T/2, y; t) = K_a(T/2 + y, t) + K_a(T/2 - y, t) = 0$$

since  $K_a(\cdot, t)$  is odd about  $T/2$ , again by Lemma 3.5. Thus for every  $y_0 \in (0, T/2)$ , the function  $x \mapsto G(x, y_0; t)$  is decreasing on  $y_0 < x < T/2$  toward the value  $G(T/2, y_0; t) = 0$ , hence it must be that  $G(x, y; t) > 0$  for all  $(x, y) \in \mathcal{R}$ , and we conclude that  $G(x, y; t) > 0$  for all  $x, y \in (0, T/2)$ . Finally, since  $G(x, y; t) > 0$  for all  $x, y \in (0, T/2)$ , we also have that  $G(x, y; t) > 0$  for all  $x, y \in (-T/2, T/2)$  since  $G$  is invariant under the maps  $x \mapsto -x$  and  $y \mapsto -y$ .  $\square$

*Remark 3.7.* Alternatively to the above proof, one can observe that for each fixed  $t > 0$ , the function  $G(x, y; t) := K_a(x - y, t) + K_a(x + y, t)$  is a solution of the IVBVP

$$\begin{cases} G_{yy} = G_{xx}, & x, y \in (-T/2, T/2) \\ G(x, 0; t) = 2K_a(x, t), & x \in (-T/2, T/2) \\ G(\pm T/2, y; t) = 0, & y \in (-T/2, T/2), \end{cases}$$

which is simply the 1D-wave equation (here treating  $y$  as the temporal variable) with homogeneous Dirichlet boundary conditions. Since  $K_a(x, t) > 0$  for  $x \in (-T/2, T/2)$ , the positivity of  $G(x, y; t)$  for  $x, y \in (-T/2, T/2)$  and  $t > 0$  follows immediately.

Turning our attention to the odd sector, similar calculations to those above yield the representation

$$(3.14) \quad e^{-\Lambda^\alpha t} f(x) = \frac{1}{2} \int_0^T [K_a(x - y, t) - K_a(x + y, t)] f(y) dy.$$

for the action of the semigroup  $e^{-\Lambda^\alpha t}$  on  $L^2_{\text{a,odd}}(0, T)$ .

**Lemma 3.8.** *For all  $t > 0$  and  $x, y \in (0, T)$ , we have*

$$K_a(x - y, t) - K_a(x + y, t) > 0.$$

*In particular, the semigroup  $e^{-\Lambda^\alpha t}$  restricted to  $L^2_{\text{a,odd}}(0, T)$  is positivity improving, i.e. if  $f \in L^2_{\text{a,odd}}(0, T)$  is nontrivial with  $f(x) \geq 0$  for  $x \in (0, T)$ , then  $e^{-\Lambda^\alpha t} f(x) > 0$  for all  $x \in (0, T)$ .*

*Proof.* Fix  $t > 0$  and  $x, y \in (0, T)$ , and observe that the  $T$ -antiperiodicity of  $K_a(\cdot, t)$  implies

$$\begin{aligned} K_a(x - y, t) - K_a(x + y, t) &= K_a(x - y, t) + K_a(x + y - T, t) \\ &= K_a\left(\left(x - \frac{T}{2}\right) - \left(y - \frac{T}{2}\right)\right) + K_a\left(\left(x - \frac{T}{2}\right) + \left(y - \frac{T}{2}\right)\right). \end{aligned}$$

Since  $x - T/2, y - T/2 \in (-T/2, T/2)$ , the proof follows by Lemma 3.6.  $\square$

From Lemma 3.6 and Lemma 3.8, the ground state eigenfunctions of  $e^{-\Lambda^\alpha t}$  acting on the invariant sectors  $L^2_{\text{a,even/odd}}(0, T)$  are positivity improving. Since the operator  $L$  defined in (3.5) is a relatively compact perturbation of  $-\Lambda^\alpha$ , we can apply standard Perron-Frobenius arguments to deduce that the largest eigenvalues of  $e^{-Lt}$  restricted to  $H^{\alpha/2}_{\text{a,even}}(0, T)$  and  $H^{\alpha/2}_{\text{a,odd}}(0, T)$  separately are simple with strictly positive eigenfunction on  $(-T/2, T/2)$  and  $(0, T)$ , respectively; see [46, Theorem XIII.44], for instance. By the functional calculus, this establishes the following characterization of the antiperiodic ground states.

**Theorem 3.9** (Antiperiodic Ground State Theory). *Let  $\alpha \in (1, 2)$  and let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be an even, smooth,  $T$ -periodic potential and consider the linear operator  $L = \Lambda^\alpha + V(x)$  acting on  $L_a^2(0, T)$ .*

- (a) *The ground state eigenvalue of  $L$  restricted to  $L_{a,\text{even}}^2(0, T)$  is simple, and the corresponding  $T$ -antiperiodic, even eigenfunction is sign-definite on  $(-T/2, T/2)$ .*
- (b) *The ground state eigenvalue of  $L$  restricted to  $L_{a,\text{odd}}^2(0, T)$  is simple, and the corresponding  $T$ -antiperiodic, odd eigenfunction is sign-definite on  $(0, T)$ .*

*Proof.* Parts (a) and (b) follow directly from [46, Theorem XIII.44], as discussed above. The ordering between the even and odd ground state eigenvalues in part (c) follows from Proposition 3.10 in Appendix A.  $\square$

While Theorem 3.9 establishes the simplicity of the even and odd antiperiodic ground state eigenvalues of  $L$  on the respective subspace, it is natural to consider the *ordering* between these ground state eigenvalues. When the potential has sufficiently small amplitude, the ordering between odd and even  $T$ -antiperiodic ground state eigenvalues given in Theorem 3.9(c) may be verified directly through the use of bifurcation theory: see, for example, Proposition 6.2 and Remark 6.3 in [44] where the analysis was carried out in a local context. In that case, the ground state eigenvalues agree at zero-amplitude and one tracks the splitting of these eigenvalues for very small amplitudes. For general amplitude potentials, however, in the local case  $\alpha = 2$  it was shown in [16, Lemma 2.2] through the use ODE techniques and increasing/decreasing rearrangement inequalities that the ordering of these ground states depends sensitively on the monotonicity properties of the periodic potential  $V$  in (3.5). Using symmetric antiperiodic rearrangement inequalities, together with the above nonlocal ground state theory, we are able to extend the results of [16] to the nonlocal setting  $\alpha \in (1, 2)$ ; see Appendix A. Such information will be used heavily in the coming sections.

**Proposition 3.10** (Ground State Ordering). *Let  $\alpha \in (1, 2)$  and let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be an even, smooth,  $T$ -periodic potential, and consider the linear operator  $L = \Lambda^\alpha + V(x)$  acting on  $L_a^2(0, T)$ .*

- (i) *If the potential  $V$  is nonincreasing on  $(0, T/2)$ , then the ground state  $T$ -antiperiodic eigenvalue of  $L$  has at least one odd eigenfunction, i.e.*

$$\min \sigma \left( L|_{L_{a,\text{odd}}^2(0, T)} \right) \leq \min \sigma \left( L|_{L_{a,\text{even}}^2(0, T)} \right).$$

- (ii) *If the potential  $V$  is nondecreasing on  $(0, T/2)$ , then the ground state  $T$ -antiperiodic eigenvalue of  $L$  has at least one even eigenfunction, i.e.*

$$\min \sigma \left( L|_{L_{a,\text{even}}^2(0, T)} \right) \leq \min \sigma \left( L|_{L_{a,\text{odd}}^2(0, T)} \right).$$

## 3.2 Antiperiodic Oscillation Theory

In addition to the above ground state theories, we require a Sturm-Liouville type oscillation theory to characterize the possible nodal patterns for the second antiperiodic eigenfunctions of the fractional Schrödinger operators  $L_\pm$ . To this end, first note that an  $H_a^{\alpha/2}(0, T)$ -eigenfunction of  $L_\pm$  is necessarily continuous, bounded, and can be chosen to be real-valued. Following the ideas in

[18] and [42], we proceed by extending the eigenvalue problem associated to  $L_{\pm}$  on  $L_a^2(0, T)$  to an appropriate local problem in the upper half space.

Note that the operator  $\Lambda^{\alpha}$  acting on  $L_a^2(0, T)$  can be viewed as the Dirichlet-to-Neumann operator for a suitable *local* problem in the antiperiodic half-strip  $[0, T] \times (0, \infty)$ . Indeed, following [11, 47], for a given  $\alpha \in (0, 2)$ , there exists a constant  $C(\alpha)$  such that for any  $f \in H_a^{\alpha}(0, T)$  we have

$$C(\alpha)\Lambda^{\alpha}f := \lim_{y \rightarrow 0^+} y^{1-\alpha} w_y(\cdot, y),$$

where  $w =: E(f) \in C^{\infty}((0, \infty); H_a^{\alpha/2}(0, T)) \cap C([0, \infty); L_a^2(0, T))$  is the unique solution to the elliptic boundary value problem

$$\begin{cases} \Delta w + \frac{1-\alpha}{y} w_y = 0, & \text{in } [0, T]_{\text{antiper}} \times (0, \infty) \\ w = f & \text{on } [0, T]_{\text{antiper}} \times \{0\} \end{cases}$$

As in [18, 42], it follows that the eigenvalue problems for  $L_{\pm}$  on  $L_a^2(0, T)$  can be extended to an eigenvalue problem for a *local* elliptic problem in the antiperiodic upper-half space  $[0, T]_{\text{antiper}} \times (0, \infty)$  and, as such, one may derive a variational characterization for the  $T$ -antiperiodic eigenvalues and eigenfunctions of  $L_{\pm}$ .

If  $v \in L_a^2(0, T)$  is an eigenfunction associated to  $L_+$ , say, then the extension  $E(v)$  belongs to  $C^0([0, T]_{\text{antiper}} \times [0, \infty))$ . Defining the zero set of  $v$  to be

$$\mathcal{N} := \{(x, y) \in [0, T]_{\text{antiper}} \times [0, \infty) : E(v)(x, y) = 0\},$$

which is clearly closed in  $[0, T]_{\text{antiper}} \times [0, \infty)$ , we define the *nodal domains* of  $E(v)$  to be the connected components of the open set  $([0, T]_{\text{antiper}} \times [0, \infty)) \setminus \mathcal{N}$ . Recalling the classical Courant nodal domain theorems yield an upper bound for the number of nodal domains of  $E(v)$  in  $[0, T]_{\text{antiper}} \times (0, \infty)$ , we find the following oscillation result.

**Lemma 3.11** (Antiperiodic Oscillation Theory). *Under the hypothesis of Theorem 3.9, any even (resp. odd)  $T$ -antiperiodic eigenfunctions of  $L$  associated with the second eigenvalue (not counting multiplicity)<sup>8</sup> has at most two sign changes over  $(-T/2, T/2)$  (resp.  $(0, T)$ ).*

*Proof.* The proof follows along the same lines as [42, Lemma 3.2] and [18, Theorem 3.1], and hence we only sketch the details here. Consider the spectrum of  $L$  acting on  $L_{a, \text{even}}^2(0, T)$ . Suppose that  $v(x)$  is an even  $T$ -antiperiodic eigenfunction of  $L$  associated with the its second eigenvalue  $\Lambda_2$ , and suppose that  $v$  has at least three sign changes in  $(-T/2, T/2)$ . Clearly, since  $v$  is even, it follows that  $v$  actually has at least four sign changes in  $(-T/2, T/2)$ , and hence there are points

$$-T/2 < x_1 < y_1 < x_2 < y_2 < x_3 < T/2$$

such that, up to switching signs,  $v(x_j) > 0$  and  $v(y_j) < 0$ . Now, using the aforementioned variational characterization of  $\Lambda_2$ , it follows from a standard Courant nodal domain argument that the extension  $E(v)$  can have at most two nodal domains in the strip  $(-T/2, T/2) \times (0, \infty)$ . Since the nodal domains are open and connected, thus pathwise connected, in  $(-T/2, T/2) \times (0, \infty)$ , we may find continuous curves  $\gamma_{\pm} \in C^0([0, 1]; [-T/2, T/2) \times [0, \infty))$  such that

$$\gamma_+(0) = x_1, \quad \gamma_+(1) = x_2, \quad \gamma_-(0) = y_1, \quad \gamma_-(1) = y_2$$

---

<sup>8</sup>That is, only the distinct elements of the  $T$ -antiperiodic spectrum of  $L$  are listed.

and

$$E(v)(\gamma_+(t)) > 0, \quad E(v)(\gamma_-(t)) < 0 \quad \text{for all } t \in [0, 1].$$

In particular,  $\gamma_+(t)$  belongs to the same nodal domain for all  $t \in (0, 1)$ , denoted  $\Omega_+$ , while  $\gamma_-(t)$  belongs to the same nodal domain  $\Omega_-$  for all  $t \in (0, 1)$ . By the Jordan curve theorem, it follows that the curves  $\gamma_\pm$  must cross at least once in  $(-T/2, T/2) \times (0, \infty)$ , yielding a contradiction. Alternatively, observe that by joining the points  $x_{1,2}$  and  $y_{1,2}$  to a point  $P$  in  $(-T/2, T/2) \times (-\infty, 0)$ , and then connecting  $x_1$  and  $y_2$  by a curve in  $(-T/2, T/2) \times (-\infty, 0)$ , one embeds the complete graph  $K_5$  in the plane. Since  $K_5$  is not planar, one again finds a contradiction, which establishes the desired oscillation estimate for the even eigenfunctions. A similar argument applies to the odd eigenfunctions.  $\square$

### 3.3 Proof of Nondegeneracy

Now that we have information regarding the  $T$ -antiperiodic ground state eigenfunctions of  $L_\pm$  and the nodal patterns for their second  $T$ -antiperiodic eigenfunctions, we aim to establish the nondegeneracy of the linearization  $\delta^2 \mathcal{E}(\phi)$ . To this end, for each  $\mu > 0$  let  $\phi(\cdot; \mu) \in H_a^{\alpha/2}(0, T)$  be a real-valued local minimizer of  $\mathcal{E}(\cdot; \mu) := \mathcal{E}(\cdot; 0, \mu)$  over  $H_a^{\alpha/2}(0, T)$  subject to fixed  $Q(u) = \mu$  and  $N(u) = 0$ . Then by construction, the second derivative test for constrained extrema yields

$$\delta^2 \mathcal{E}(\phi)|_{\{\delta Q(\phi), \delta N(\phi)\}^\perp} \geq 0,$$

where here

$$\{\delta Q(\phi), \delta N(\phi)\}^\perp := \left\{ h \in H_a^{\alpha/2}([0, T]; \mathbb{C}) : \langle \phi, h \rangle = \langle i\phi', h \rangle = 0 \right\}$$

denotes the tangent space at  $\phi$  to the codimension two constrained subspace

$$\Sigma_\mu := \left\{ \psi \in H_a^{\alpha/2}(0, T) : Q(\psi) = \mu, \quad N(\psi) = 0 \right\}$$

in  $H_a^{\alpha/2}(0, T)$ . Recall that the inner product  $\langle \cdot, \cdot \rangle$  is defined throughout as

$$\langle u, v \rangle = \operatorname{Re} \int_0^T u \bar{v} \, dx.$$

By Courant's mini-max principle, this implies that the operator  $\delta^2 \mathcal{E}(\phi)$  has at most two negative  $T$ -antiperiodic eigenvalues. Specifically, since  $\delta Q(\phi) = \phi$  and  $\delta N(\phi) = i\phi'$  are real and imaginary valued, respectively, it follows that the linear operators  $L_+$  and  $L_-$  each have at most one negative  $T$ -antiperiodic eigenvalue, with

$$L_+|_{\{\delta Q(\phi)\}^\perp} \geq 0 \quad \text{and} \quad L_-|_{\{\operatorname{Im}(\delta N(\phi))\}^\perp} \geq 0.$$

A finer description of the spectral properties of  $L_\pm$  is given below.

**Lemma 3.12.** *Under the hypothesis of Proposition 3.1, the following are true:*

(i) *The operator  $\delta^2 \mathcal{E}(\phi)$  acting on  $L_a^2(0, T)$  has at most one negative eigenvalue, with*

$$L_+ \geq 0 \quad \text{and} \quad n_-(L_-) \leq 1.$$

- (ii)  $\phi' \in \ker(L_+)$ , and it corresponds to the ground state eigenfunction of  $L_+$  restricted to the sector of odd functions in  $H_a^{\alpha/2}(0, T)$ .
- (iii)  $\phi \in \ker(L_-)$ , and it corresponds to the ground state eigenfunction of  $L_-$  restricted to the sector of even functions in  $H_a^{\alpha/2}(0, T)$ .
- (iv) The functions  $\phi'$  and  $\phi^{2\sigma}\phi'$  are in the range of  $L_-$ .
- (v) The function  $\phi^{2\sigma+1}$  is in the range of  $L_+$ .

*Proof.* First, note that (2.1) is equivalent to  $L_- \phi = 0$ , while differentiating the profile equation with respect to  $x$  gives  $L_+ \phi' = 0$ . Claims (ii) and (iii) now follow immediately from the ground state theory in Theorem 3.9 and the monotonicity properties of  $\phi$  guaranteed by Lemma 2.2. Moreover, by the ordering of the even and odd  $T$ -antiperiodic ground states of  $L_{\pm}$  given by part (c) of Theorem 3.9, the fact that  $\phi$  and  $\phi'$  are even and odd, respectively, implies (i) by the above discussion.

Next, observing that  $L_+ = L_- + 2\sigma\phi^{2\sigma}$ , the identities  $L_+\phi' = 0$  and  $L_-\phi = 0$  immediately imply that  $L_-\phi' = -2\sigma\phi^{2\sigma}\phi'$  and  $L_+\phi = 2\sigma\phi^{2\sigma+1}$ . Finally, differentiating the profile equation (2.1) with respect to  $c$  at  $c = 0$  gives

$$L_+ \operatorname{Re} \left( \frac{\partial \phi}{\partial c} \right) = - \left( \frac{\partial \omega}{\partial c} \Big|_{c=0} \right) \phi, \quad L_- \operatorname{Im} \left( \frac{\partial \phi}{\partial c} \right) = -\phi'.$$

The first equation above obviously holds thanks to Lemma 2.2, while the second shows that  $\phi'$  is in the range of  $L_-$ . This establishes (iv) and (v).  $\square$

*Remark 3.13.* In the proof that  $\phi'$  lies in the range of  $L_-$  above, we heavily relied on the fact that the real-valued,  $T$ -antiperiodic standing profile  $\phi(\cdot; \mu)$  is a member of a more general family of complex-valued  $T$ -antiperiodic traveling waves  $\phi(\cdot; c, \mu)$  defined for  $|c|$  sufficiently small; see Proposition 2.1 and Lemma 2.2. In the local case  $\alpha = 2$ , one may of course rely on the Galilean invariance of (1.1) to produce such a curve of traveling solutions near  $c = 0$ , and differentiating along this curve yields the same result. Since  $\alpha \in (1, 2)$ , such an (exact) Galilean invariance does not exist, which is why we had to take extra care in our existence theory to carefully construct such a curve of solutions for a given  $\mu > 0$ .

With the above preliminaries in mind, we can now establish the main result of this section.

*Proof of Proposition 3.1.* First, note that since  $\phi^{2\sigma}$  is even and  $T$ -periodic by construction, the sectors  $L_{a,\text{odd}}^2(0, T)$  and  $L_{a,\text{odd}}^2(0, T)$  of even and odd, respectively,  $T$ -antiperiodic functions are invariant subspaces of the operators  $L_{\pm}$ . In particular, the operators  $L_{\pm}$  respect the orthogonal decomposition

$$L_a^2(0, T) = L_{a,\text{odd}}^2(0, T) \oplus L_a^2(0, T)_{\text{even}}$$

so that

$$\sigma \left( L_{\pm} \Big|_{L_a^2(0, T)} \right) = \sigma \left( L_{\pm} \Big|_{L_{a,\text{odd}}^2(0, T)} \right) \cup \sigma \left( L_{\pm} \Big|_{L_{a,\text{even}}^2(0, T)} \right).$$

Since Lemma 3.12 implies that

$$\ker \left( L_+ \Big|_{L_{a,\text{odd}}^2(0, T)} \right) = \operatorname{span} \{ \phi' \} \quad \text{and} \quad \ker \left( L_- \Big|_{L_{a,\text{even}}^2(0, T)} \right) = \operatorname{span} \{ \phi \},$$

it remains to verify that  $\ker \left( L_+|_{L_{a,\text{even}}^2(0,T)} \right)$  and  $\ker \left( L_-|_{L_{a,\text{odd}}^2(0,T)} \right)$  are trivial.

First, suppose there exists a non-trivial solution  $v \in L_{a,\text{even}}^2(0,T)$  of the equation  $L_+v = 0$ . Since  $L_+$  is self adjoint on  $L_a^2(0,T)$ , the Fredholm alternative implies that  $v$  must be orthogonal to the range of the operator  $L_+$  acting on  $L_a^2(0,T)$ . Since zero is the ground state eigenvalue of  $L_+$  acting on  $L_a^2(0,T)$  by Lemma 3.12(i), it follows that  $v$  is the even ground state eigenfunction for  $L_+$  and hence, by Theorem 3.9, may be chosen to be strictly positive on  $(-T/2, T/2)$ . To reach the desired contradiction, observe that Lemma 3.12(v) implies the function  $\phi^{2\sigma+1}$  is in the range of  $L_+$  acting on  $L_a^2(0,T)$ . Since  $\phi$  is positive on  $(-T/2, T/2)$  we have

$$\int_0^T v(x)\phi^{2\sigma+1}(x)dx \neq 0,$$

contradicting the Fredholm alternative. Consequently,  $\ker(L_+|_{L_{a,\text{even}}^2(0,T)}) = \{0\}$  and hence

$$(3.15) \quad \ker(L_+) = \text{span}\{\phi\},$$

verifying the nondegeneracy of  $L_+$  on  $L_a^2(0,T)$ .

Next, we turn our attention to  $L_-$ . First, we claim that  $L_-$  has exactly one negative  $T$ -antiperiodic eigenvalue. To this end, suppose (to show a contradiction) that  $L_-$  does not have a negative eigenvalue. Since  $L_-\phi = 0$  and  $\phi$  is even, the ground state ordering (Theorem 3.9 (c)) implies that there exists  $\psi \in L_{a,\text{odd}}^2(0,T)$  such that  $L_-\psi = 0$  with  $\psi$  being the ground state on  $L_-|_{L_{a,\text{odd}}^2(0,T)}$ . Then by Theorem 3.9 (b),  $\psi$  is sign-definite on  $(0,T)$ , hence  $\langle \psi, \phi' \rangle \neq 0$ , i.e.  $\phi$  is not orthogonal to  $\phi'$ . But  $\phi' \in \text{range}\left(L_-|_{L_{a,\text{odd}}^2(0,T)}\right) = \ker\left(L_-|_{L_{a,\text{odd}}^2(0,T)}\right)^\perp$ , a contradiction of the Fredholm alternative. Thus it must be that  $n_-(L_-) \geq 1$ , which implies that  $n_-(L_-) = 1$  by Lemma 3.12 (i). Now, since  $L_-\phi = 0$ , it follows that  $\lambda = 0$  is the second eigenvalue of  $L_-$  acting on  $L_a^2(0,T)$ . As above, suppose there exists a nontrivial solution  $v \in L_{a,\text{odd}}^2(0,T)$  to the equation  $L_-v = 0$ . By Lemma 3.11,  $v$  may change signs at most twice on  $(0,T)$ . We will show that such a nontrivial  $v$  cannot exist by again using the Fredholm alternative. To this end, note that if  $v$  has a fixed sign on  $(0,T)$ , then using that  $\phi' < 0$  on  $(0,T)$  we have

$$\int_0^T v(x)\phi'(x) dx \neq 0,$$

which contradicts the Fredholm alternative since  $\phi' \in \text{range}(L_-)$  by Lemma 3.12(iv). Thus, any non-trivial  $v \in \ker(L_-)$  must change signs at least once in  $(0,T)$  and, since odd  $T$ -antiperiodic functions are even about  $x = T/2$ , such a function must have exactly two sign changes in  $(0,T)$ ; one at some  $x = x_0 \in (0, T/2)$  and the other at  $x = T - x_0 \in (T/2, T)$ . Define

$$\eta(x) := \phi'(x) (\phi(x)^{2\sigma} - \phi(x_0)^{2\sigma})$$

and note that  $\eta \in \text{range}(L_-)$  by Lemma 3.12(iv) and, further,  $\eta$  changes signs at  $x = x_0$  and  $x = T - x_0$ , by monotonicity of  $\phi$  on  $(0,T)$ . Consequently,  $\int_0^T \eta(x)v(x)dx \neq 0$  again contradicting the Fredholm alternative. Thus, it must be that  $\ker\left(L_-|_{L_{a,\text{odd}}^2(0,T)}\right) = \{0\}$ , and we conclude that

$$(3.16) \quad \ker(L_-) = \text{span}\{\phi'\},$$

verifying the nondegeneracy of  $L_-$  on  $L_a^2(0, T)$ .

Finally, since  $\mathcal{L} = \text{diag}(L_+, L_-)$  is diagonal, we have by (3.15) and (3.16) that

$$\ker(\mathcal{L}) = \text{span} \left\{ \begin{pmatrix} \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right\},$$

establishing Proposition 3.1.  $\square$

Before proceeding, we point out an interesting observation regarding the parameterization of the profiles  $\phi(\cdot; c, \mu)$  in a neighborhood of a given  $(c, \mu) = (0, \mu_0)$  with  $\mu_0 > 0$ . From the view point of special solutions to the given PDE, one might prefer that the profiles be parameterized the Lagrange multiplier  $\omega$  instead of by the  $L^2$ -norm of the solution. By the Implicit Function Theorem, the manifold of nearby profiles  $\phi(\cdot; c, \mu)$  can be reparameterized in a  $C^1$  manner by the parameters  $(c, \omega)$  near  $(c, \mu) = (0, \mu_0)$  provided the Jacobian matrix

$$(3.17) \quad \frac{\partial(c, \omega)}{\partial(c, \mu)} = \begin{pmatrix} 1 & 0 \\ \frac{\partial \omega}{\partial c} & \frac{\partial \omega}{\partial \mu} \end{pmatrix}$$

is non-singular at  $(c, \mu) = (0, \mu_0)$  or, equivalently, if  $\frac{\partial \omega}{\partial \mu}(0, \mu_0) \neq 0$ . To see that the above Jacobian is indeed non-singular, observe that by differentiating the profile equation (2.1) with respect to  $\mu$  at  $c = 0$  yields the identity

$$L_+ \left( \frac{\partial \phi}{\partial \mu} \right) = - \left( \frac{\partial \omega}{\partial \mu} \right) \phi.$$

Thus, if it were the case that  $\frac{\partial \omega}{\partial \mu} = 0$  at  $(c, \mu) = (0, \mu_0)$ , then by Proposition 3.1 it must be that  $\frac{\partial \phi}{\partial \mu} = A\phi'$  for some constant  $A \in \mathbb{R}$ , which is impossible since  $\frac{\partial \phi}{\partial \mu}$  is even, and  $\phi'$  is odd. As a result, the nondegeneracy result Proposition 3.1 implies that the Jacobian matrix (3.17) is necessarily non-singular, and hence, if desired, we could consider the real-valued solutions constructed in Lemma 2.2 to be parameterized by the waves speed  $c$  and the temporal frequency (Lagrange multiplier)  $\omega$ .

However, while the parameterization by  $(c, \omega)$  may seem more natural mathematically, especially from the perspective of obtaining explicit formulae, it is actually more natural from the standpoint of *local dynamics* to attempt to reparameterize the family  $\phi(\cdot; c, \mu)$  completely in terms of the conserved quantities of the PDE flow generated by (1.1). Indeed, it has been observed in several different contexts that a key ingredient in understanding the local dynamics near special solutions, for example traveling waves, is that, locally, the manifold of special solutions can be parameterized by the conserved quantities of the PDE flow. In the case of Hamiltonian evolutionary PDE, this typically arises as a “ $\frac{dP}{dc} \neq 0$ ” type condition, where the  $P$  is a charge functional associated via Noether’s theorem to continuous Lie point symmetry of the equation: see [45, 6, 5, 40] in the case of traveling solitary waves, and [20, 42, 32, 9, 8] in the case of periodic traveling waves, while this observation has likewise been made in the context of partially dissipative systems of conservation or balance laws in [33]. In the present context, the profiles  $\phi(\cdot; c, \mu)$  may be locally reparameterized in a  $C^1$  manner by the conserved quantities  $(N, Q)$  provided that the Jacobian matrix

$$(3.18) \quad \frac{\partial(N, Q)}{\partial(c, \mu)} = \begin{pmatrix} \frac{\partial N}{\partial c} & \frac{\partial N}{\partial \mu} \\ 0 & 1 \end{pmatrix}$$

is non-singular at a given  $(c, \mu)$  or, equivalently, that  $\frac{\partial N}{\partial c} \neq 0$  at the given wave. As seen in Theorem 4.1 below, the condition  $\frac{\partial N}{\partial c} \neq 0$  ensures the nonlinear orbital stability of the waves  $\phi(\cdot; c, \mu)$  constructed here.



We end this section by demonstrating that the non-singularity of the Jacobian matrix (3.18) ensures that the generalized  $L_a^2(0, T)$ -kernel of the linearized operator associated with (1.1) supports a Jordan structure, which plays a central role in the forthcoming stability analysis. Note that linearizing (1.1) about  $\phi(\cdot; \mu)$  yields the linear system

$$v_t = -i\delta^2 \mathcal{E}(\phi)v,$$

which, by taking the Laplace transform in time and decomposing into real and imaginary parts, leads one to the spectral problem

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} \begin{pmatrix} \operatorname{Re}(v) \\ \operatorname{Im}(v) \end{pmatrix} = \lambda \begin{pmatrix} \operatorname{Re}(v) \\ \operatorname{Im}(v) \end{pmatrix}.$$

**Proposition 3.14** (Jordan Block Structure). *Under the hypotheses of Proposition 3.1, zero is  $T$ -antiperiodic generalized eigenvalue of the linearized operator*

$$J\mathcal{L} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

associated to the profile  $\phi(\cdot; c = 0, \mu)$  with algebraic multiplicity four and geometric multiplicity two provided that  $\frac{\partial N}{\partial c}(\phi(\cdot; c, \mu))$  is non-zero at  $c = 0$ .

*Proof.* By Proposition 3.1, we already know that zero is a  $T$ -antiperiodic eigenvalue of both  $L_\pm$  with algebraic multiplicity at least one and geometric multiplicity precisely one. Since the skew-adjoint operator  $J$  is invertible, it follows from Lemma 3.12 that the operator  $J\mathcal{L}$  has zero as a generalized eigenvalue with geometric multiplicity two and algebraic multiplicity at least four with

$$\ker(J\mathcal{L}) = \operatorname{span} \left\{ \begin{pmatrix} \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right\} \subset \operatorname{range}(J\mathcal{L})$$

and

$$(3.19) \quad J\mathcal{L} \begin{pmatrix} 0 \\ \operatorname{Im} \left( \frac{\partial \phi}{\partial c} \right) \end{pmatrix} = \begin{pmatrix} -\phi' \\ 0 \end{pmatrix}, \quad J\mathcal{L} \begin{pmatrix} \frac{\partial \phi}{\partial \mu} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial \omega}{\partial \mu} \\ \phi \end{pmatrix}.$$

By the Fredholm alternative, the above Jordan chains terminate at height two provided that the functions  $\begin{pmatrix} 0 \\ \operatorname{Im} \left( \frac{\partial \phi}{\partial c} \right) \end{pmatrix}$  and  $\begin{pmatrix} \frac{\partial \phi}{\partial \mu} \\ 0 \end{pmatrix}$  are orthogonal to  $\ker(J\mathcal{L}) = J\ker(\mathcal{L})$ , i.e. provided that

$$\int_0^T \phi' \operatorname{Im} \left( \frac{\partial \phi}{\partial c} \right) dx = \frac{\partial N}{\partial c}(\phi) \neq 0 \quad \text{and} \quad \int_0^T \phi \frac{\partial \phi}{\partial \mu} dx = \frac{\partial Q}{\partial \mu}(\phi) \neq 0.$$

Since  $Q(\phi) = \mu$ , it follows that zero has algebraic multiplicity exactly four so long as  $\frac{\partial N}{\partial c} \neq 0$ , as claimed.  $\square$

## 4 Stability of Constrained Energy Minimizers

Let  $T > 0$  and  $\mu_0 > 0$  be fixed and let  $\phi_0 := \phi(\cdot; \mu_0)$  denote a real-valued,  $T$ -antiperiodic solution of the nonlocal profile equation (2.1) with  $c = 0$  satisfying  $Q(\phi_0) = \mu_0$ , whose existence is guaranteed by Proposition 2.1 and Lemma 2.2. The profile  $\phi_0$  is thus an equilibrium solution of the PDE

$$(4.1) \quad iu_t - \omega_0 u - \Lambda^\alpha u - |u|^{2\sigma} u = 0,$$

where here  $\omega_0 := \omega(0, \mu_0)$ . In this section, we wish to consider the stability of  $\phi_0$  under the evolution of (4.1) to general complex-valued,  $T$ -antiperiodic perturbations or, equivalently, the stability of the standing wave solution  $u(x, t; \mu_0) = e^{i\omega_0 t} \phi_0(x)$  under the evolution of (1.1) to such perturbations.

For  $\alpha > 1$  and  $\sigma > 0$ , an iteration argument reveals that the Cauchy problem for (4.1) is locally in time well-posed in  $H_a^{1/2+}([0, T]; \mathbb{C})$ . Furthermore, using conservation laws these local solutions can be extended to global ones in  $H_a^{\alpha/2}([0, T]; \mathbb{C})$  provided the initial data is in  $H_a^{\alpha/2}(0, T)$ . Throughout our analysis we work on an appropriate subspace  $X$  of  $H_a^{\alpha/2}([0, T]; \mathbb{C})$  where the Cauchy problem associated with (3.1) is locally well-posed and where the functionals  $\mathcal{H}, Q, N : X \rightarrow \mathbb{R}$  are smooth.

Observe that the evolution defined by (1.1), and hence of (4.1), is invariant under a two-parameter group of symmetries generated by spatial translations and unitary phase rotations. For each  $\phi \in X$  this motivates us to define the group orbit

$$\mathcal{O}_\phi := \left\{ e^{i\beta} \phi(\cdot - x_0) : (\beta, x_0) \in \mathbb{R}^2 \right\} \subset X.$$

Loosely speaking, we say that the standing wave  $\phi_0(\cdot; \mu_0)$  is *orbitally stable* if the group orbit  $\mathcal{O}_{\phi_0}$  is stable under the evolution of (4.1), i.e. if solutions of (4.1) remain close in the  $X$ -norm to  $\mathcal{O}_{\phi_0}$  for all future times provided their initial data is sufficiently close in the  $X$ -norm to  $\mathcal{O}_{\phi_0}$ . We elaborate further below.

Our treatment of the orbital stability problem is inspired by the Lyapunov method. Setting

$$(4.2) \quad \mathcal{E}_0(u) := \mathcal{H}(u) + \omega_0 Q(u),$$

we recall from Lemma 2.2 that  $\phi_0$  is a critical point of  $\mathcal{E}_0$ , i.e. that  $\delta \mathcal{E}_0(\phi) = 0$ , or, equivalently, that  $\phi_0$  is a critical point of  $\mathcal{H}$  subject to fixed  $Q(u) = \mu_0$  and  $N(u) = 0$ . Furthermore, Proposition 3.1 implies that the kernel of the Hessian  $\delta^2 \mathcal{E}_0(\phi_0)$  is generated by the translation and phase rotation symmetries. Intuitively, we expect the group orbit of  $\phi_0$  to be stable provided the operator  $\delta^2 \mathcal{E}_0(\phi_0)$  is convex (in an appropriate sense) at  $\phi_0$ . Below, we shall demonstrate this convexity and hence establish the orbital stability of the standing wave  $\phi_0$  under the evolution of (4.1). We now state the main result of this section.

**Theorem 4.1** (Orbital Stability). *Suppose  $\alpha \in (1, 2]$ , let  $\phi(\cdot; \mu_0) \in H_a^{\alpha/2}(0, T)$  be a real-valued,  $T$ -antiperiodic local minimizer of  $\mathcal{H}$  subject to  $Q(u) = \mu_0$  and  $N(u) = 0$  as constructed in Lemma 2.2, and suppose that*

$$(4.3) \quad \partial_c N(\phi(\cdot; c, \mu_0)) \neq 0 \quad \text{at} \quad c = 0.$$

Then for all  $\varepsilon > 0$  sufficiently small, there exists a constant  $C = C(\varepsilon)$  such that if  $v \in X$  with  $\|v\|_X \leq \varepsilon$  and  $N(\phi(\cdot; \mu_0) + v) = 0$ , and if  $u(\cdot, t)$  is a local in time solution of (4.1) with initial data  $u(\cdot, 0) = \phi_0 + v$ , then  $u(\cdot, t)$  can be continued to a solution for all  $t > 0$  and

$$\sup_{t>0} \inf_{(\beta, x_0) \in \mathbb{R}^2} \left\| u(\cdot, t) - e^{i\beta} \phi_0(\cdot - x_0) \right\|_X \leq C \|v\|_X.$$

To begin the proof of Theorem 4.1, observe Proposition 3.1 and Lemma 3.12 implies that  $\phi_0$  is a degenerate saddle point of  $\delta^2 \mathcal{E}_0$  acting on  $H_a^{\alpha/2}(0, T)$ , with one negative direction and two neutral directions. To handle these potentially unstable directions, we note that the evolution of (4.1) does not occur on the whole space  $X$ , but rather on the codimension two nonlinear manifold

$$\Sigma_0 := \{u \in X : Q(u) = \mu_0, \quad N(u) = 0\}.$$

In particular,  $\Sigma_0$  is an invariant set under the flow of (4.1), with  $\mathcal{O}_{\phi_0} \subset \Sigma_0$ . The key step in the proof of Theorem 4.1, we establish the coercivity of  $\mathcal{E}_0$  on  $\Sigma_0$  in a neighborhood of  $\mathcal{O}_{\phi_0}$  provided that condition (4.3) holds. To this end, we define

$$(4.4) \quad \mathcal{T}_0 := \text{span}\{\delta Q(\phi_0), \delta N(\phi_0)\}^\perp = \text{span}\{\phi_0, i\phi'_0\}^\perp$$

to be the tangent space in  $X$  to the submanifold  $\Sigma_0$  at  $\phi_0$ , and establish the following technical result.

**Lemma 4.2.** *Under the hypothesis of Theorem 4.1,*

$$\inf \{ \langle \delta^2 \mathcal{E}_0(\phi_0)v, v \rangle : \|v\|_X = 1, \quad v \in \mathcal{T}_0, \quad v \perp \text{span}\{\phi'_0, i\phi_0\} \} > 0.$$

*In particular, there exists a constant  $C > 0$  such that*

$$\langle \delta^2 \mathcal{E}_0(\phi)v, v \rangle \geq C \|v\|_X^2$$

*for all  $v \in \mathcal{T}_0$  with  $v \perp \text{span}\{\phi'_0, i\phi_0\}$ .*

*Proof.* First, let  $\Pi : L_a^2(0, T) \rightarrow \mathcal{T}_0$  be the self-adjoint projection onto  $\mathcal{T}_0$  and note consider the constrained operator

$$\Pi \delta^2 \mathcal{E}_0(\phi_0) : \mathcal{T}_0 \subset X \rightarrow \mathcal{T}_0.$$

By construction,  $\mathcal{T}_0$  is an invariant subspace of  $\Pi \delta^2 \mathcal{E}_0(\phi_0)$  and, further, the operator  $\Pi \delta^2 \mathcal{E}_0(\phi_0)$  is self adjoint acting on  $\mathcal{T}_0$ . Owing to the periodic boundary conditions, the spectrum of the operator  $\Pi \delta^2 \mathcal{E}_0(\phi_0)$  is real and purely discrete, consisting of infinitely many isolated eigenvalues with no finite accumulation point. Further, since  $\phi_0$  locally minimizes  $\mathcal{E}_0$  with fixed  $Q(u) = \mu_0$  and  $N(u) = 0$ , we have that

$$\Pi \delta^2 \mathcal{E}_0(\phi_0) \geq 0.$$

It follows that  $\Pi \delta^2 \mathcal{E}(\phi_0)$  is positive-semidefinite acting on  $\mathcal{T}_0$ , and hence the eigenvalues of  $\Pi \delta^2 \mathcal{E}_0(\phi_0)$  may be listed, not counting multiplicity, as

$$\Lambda_1 = 0 < \Lambda_2 < \Lambda_3 < \Lambda_4 < \dots \rightarrow +\infty.$$

Thanks to the spectral gap between  $\Lambda_1$  and  $\Lambda_2$ , if we now define the self-adjoint spectral projection

$$\Pi_0 : \mathcal{T}_0 \mapsto \ker(\Pi \delta^2 \mathcal{E}_0(\phi_0)) \cap \mathcal{T}_0$$

it follows that

$$\sigma((1 - \Pi_0)\Pi\delta^2\mathcal{E}_0(\phi_0)) = \sigma(\Pi\delta^2\mathcal{E}_0(\phi_0)) \setminus \{0\}$$

so that, in particular,

$$\inf \{ \langle \delta^2\mathcal{E}(\phi)v, v \rangle : \|v\|_X = 1, v \in \mathcal{T}_0, v \perp \ker(\Pi\delta^2\mathcal{E}_0(\phi_0)) \} = \Lambda_2 > 0.$$

This immediately provides the estimate

$$\langle \Pi\delta^2\mathcal{E}_0(\phi_0)v, v \rangle \geq \Lambda_2 \|v\|_{L^2(0,T)}^2$$

for all  $v \in \mathcal{T}_0$  with  $v \perp \ker(\Pi\delta^2\mathcal{E}_0(\phi_0))$ . By the definition of the space  $X$ , it is now easy to see this in turn implies the estimate

$$\langle \Pi\delta^2\mathcal{E}_0(\phi_0)v, v \rangle \geq \Lambda_2 \|v\|_X^2$$

for all such  $v$ ; see [35, Lemma 5.2.3].

By Proposition 3.1, the desired result follows immediately provided

$$(4.5) \quad \ker(\Pi\delta^2\mathcal{E}_0(\phi_0)) = \ker(\delta^2\mathcal{E}_0(\phi_0)) = \text{span}\{\phi'_0, i\phi_0\}.$$

Well, if  $\psi \in \mathcal{T}_0$  lies in the kernel of  $\Pi\delta^2\mathcal{E}_0(\phi_0)$  then, by the definition of the projection  $\Pi$ , we have

$$\delta^2\mathcal{E}_0(\phi_0)\psi = A\phi_0 + iB\phi'_0$$

for some constants  $A, B \in \mathbb{R}$ . From Proposition 3.1 and (3.19), it follows that all solutions of the above equation are of the form

$$\psi = -A \left( \frac{\partial\omega}{\partial\mu} \right)^{-1} \frac{\partial\phi_0}{\partial\mu} - iB \text{Im} \left( \frac{\partial\phi_0}{\partial c} \right) + \gamma_1 \phi'_0 + i\gamma_2 \phi_0$$

for some constants  $\gamma_1, \gamma_2 \in \mathbb{R}$ . The requirement that  $\psi \in \mathcal{T}_0$  now enforces

$$\langle \phi_0, \psi \rangle = -A \left( \frac{\partial\omega}{\partial\mu} \right)^{-1} \int_0^T \frac{\partial\phi_0}{\partial\mu} \phi_0 \, dx = 0$$

and

$$\langle i\phi'_0, \psi \rangle = -B \int_0^T \phi'_0 \text{Im} \left( \frac{\partial\phi_0}{\partial c} \right) dx = 0.$$

Since  $\partial_\mu Q(\phi_0(\cdot; c, \mu)) = 1$  for all  $\mu > 0$ , and  $\partial_c N(\phi_0(\cdot; c, \mu)) \neq 0$  at  $(c, \mu) = (0, \mu_0)$  by assumption, it follows that while the operator  $\Pi\delta^2\mathcal{E}_0(\phi_0)$  formally sends the functions  $\frac{\partial\phi_0}{\partial\mu}$  and  $\frac{\partial\phi_0}{\partial c}$  to zero, these functions do not lie in the admissible space  $\mathcal{T}_0$  under the given hypotheses. In particular, it follows that  $A = B = 0$  above, and hence that (4.5) holds, completing the proof.  $\square$

*Remark 4.3.* Alternatively to the direct proof above, one may use an index formula to verify the constrained kernel condition (4.5). Indeed, observing that since  $\phi_0, i\phi'_0 \in \ker(\delta^2\mathcal{E}_0(\phi_0))^\perp$  and that  $(\delta^2\mathcal{E}_0(\phi_0))^{-1} \phi_0 = -\left(\frac{\partial\omega}{\partial\mu}\right)^{-1} \frac{\partial\phi_0}{\partial\mu}$ , and  $(\delta^2\mathcal{E}_0(\phi_0))^{-1} (i\phi'_0) = -\frac{\partial\phi_0}{\partial c}$  it follows that

$$(4.6) \quad \dim(\ker(\Pi\delta^2\mathcal{E}_0(\phi_0))) = \dim(\ker(\delta^2\mathcal{E}_0(\phi_0))) + z\left(\frac{\partial(N, Q)}{\partial(c, \mu)} \Big|_{(c, \mu)=(0, \mu_0)}\right)$$

where  $z(D)$  denotes the number of zero eigenvalues of a given matrix  $D$ ; see, for instance, [34, Theorem 2.1]. The above Jacobian has already been computed in (3.18) and shown to be non-singular under the condition that  $\frac{\partial N}{\partial c} \neq 0$  at  $\phi_0$ , yielding the equivalence of the kernels in (4.5)

Next, we introduce the semidistance  $\rho$  on  $X$  defined via

$$\rho(u, v) := \inf_{(\beta, x_0) \in \mathbb{R}^2} \left\| u - e^{i\beta} v(\cdot - x_0) \right\|_X,$$

and observe that  $\rho(u, v)$  simply measures the distance in  $X$  from  $u$  to the group orbit  $\mathcal{O}_v$  or, equivalently, from  $v$  to  $\mathcal{O}_u$ . Next, we show that the functional  $\mathcal{E}_0$  is coercive on the nonlinear manifold  $\Sigma_0$  with respect to the semidistance  $\rho$ .

**Proposition 4.4** (Coercivity). *Under the hypothesis of Theorem 4.1, there exist constants  $\varepsilon > 0$  and  $C = C(\varepsilon) > 0$  such that if  $u \in \Sigma_0$  with  $\rho(u, \phi_0) < \varepsilon$ , then*

$$\mathcal{E}_0(u) - \mathcal{E}_0(\phi_0) \geq C\rho(u, \phi_0)^2.$$

*Proof.* By the implicit function theorem, for  $\varepsilon > 0$  sufficiently small there exists a neighborhood  $\mathcal{U}_\varepsilon := \{u \in X : \rho(u, \phi_0) < \varepsilon\}$  of  $\mathcal{O}_{\phi_0}$  and continuously differentiable maps  $\tau, \beta : \mathcal{U}_\varepsilon \rightarrow \mathbb{R}$  such that

$$(4.7) \quad \tau(\phi_0) = 0, \quad \beta(\phi_0) = 0, \quad \left\langle e^{i\beta(u)} u(\cdot + \tau(u)), \phi'_0 \right\rangle = 0, \quad \text{and} \quad \left\langle e^{i\beta(u)} u(\cdot + \tau(u)), i\phi_0 \right\rangle = 0$$

for all  $u \in \mathcal{U}_\varepsilon$ . Since  $\mathcal{E}_0$  is invariant under spatial translations, it will suffice to show that  $\mathcal{E}_0(u(\cdot + \tau(u)) - \mathcal{E}_0(\phi_0) \geq C\rho(u(\cdot + \tau(u)), \phi_0)^2$ . Now, fix  $u \in \mathcal{U}_\varepsilon \cap \mathcal{T}_0$  and note we can write

$$e^{i\beta(u)} u(\cdot + \tau(u)) = \phi_0 + C_1 \phi_0 + C_2 \phi'_0 + y,$$

where the  $C_1, C_2 \in \mathbb{C}$  and  $y \in \mathcal{T}_0$  with  $y \perp \ker(\delta^2 \mathcal{E}(\phi_0))$ . In particular, note that  $C_1 = C_2 = y = 0$  at  $u = \phi_0$ . Furthermore, under the above decomposition the orthogonality conditions in (4.7) reduce to

$$\operatorname{Re}(C_2) \int_0^T (\phi'_0)^2 dx = \operatorname{Im}(C_1) \int_0^T \phi_0^2 dx = 0,$$

and hence  $C_1 \in \mathbb{R}$  and  $C_2 = iC_3$  for some  $C_3 \in \mathbb{R}$ .

Next, set

$$h := e^{i\beta(u)} u(\cdot + \tau(u)) - \phi = C_1 \phi_0 + iC_3 \phi'_0 + y$$

and note, after possibly translating and rotating  $\phi_0$ , we may assume without loss of generality that  $\|h\|_X < \varepsilon$ . Since the functionals  $Q$  and  $N$  are left invariant by spatial translations and unitary phase rotations, Taylor's theorem yields

$$\begin{cases} Q(u) = Q(e^{i\beta(u)} u(\cdot + \tau(u))) = Q(\phi_0) + \langle \delta Q(\phi_0), h \rangle + \mathcal{O}(\|h\|_X^2) \\ N(u) = N(e^{i\beta(u)} u(\cdot + \tau(u))) = N(\phi_0) + \langle \delta N(\phi_0), h \rangle + \mathcal{O}(\|h\|_X^2) \end{cases}$$

so that, since  $u \in \Sigma_0$ ,

$$C_1 \|\phi_0\|_X^2 = \langle \delta Q(\phi_0), h \rangle = \mathcal{O}(\|h\|_X^2) \quad \text{and} \quad C_3 \|\phi'_0\|_X^2 = \langle \delta N(\phi_0), h \rangle = \mathcal{O}(\|h\|_X^2).$$

Consequently,  $C_1, C_3 = \mathcal{O}(\|h\|_X^2)$ .

Finally, using Taylor's theorem again, we find

$$\mathcal{E}_0(u) = \mathcal{E}_0(e^{i\beta(u)} u(\cdot + \tau(u))) = \mathcal{E}_0(\phi_0) + \langle \delta \mathcal{E}_0(\phi_0), h \rangle + \frac{1}{2} \langle \delta^2 \mathcal{E}_0(\phi_0) h, h \rangle + o(\|h\|_X^2)$$

so that

$$\begin{aligned}\mathcal{E}_0(u) - \mathcal{E}_0(\phi_0) &= \frac{1}{2} \langle \delta^2 \mathcal{E}_0(\phi_0) h, h \rangle + o(\|h\|_X^2) \\ &= \frac{1}{2} \langle \delta^2 \mathcal{E}_0(\phi_0) y, y \rangle + \mathcal{O}(C_1^2 + C_3^2) + \mathcal{O}((|C_1| + |C_3|)\|h\|_X) + o(\|h\|_X^2) \\ &= \frac{1}{2} \langle \delta^2 \mathcal{E}_0(\phi_0) y, y \rangle + o(\|h\|_X^2).\end{aligned}$$

where the last equality is justified by the above estimates on  $C_1, C_3$ . Since  $y \in \mathcal{T}_0$  with  $y \perp \text{span}\{\phi'_0, i\phi_0\}$ , it follows by Lemma 4.2 that

$$\langle \delta^2 \mathcal{E}_0(\phi_0) y, y \rangle \geq C \|y\|_X^2$$

for some constant  $C > 0$ . Since the estimates  $C_1, C_3 = \mathcal{O}(\|h\|_X^2)$  yield

$$\|y\|_X = \|h - C_1\phi - iC_3\phi\|_X \geq \|h\|_X - C^* \|h\|_X^2$$

for some constant  $C^* > 0$ , it follows from the definition of  $h$  that for  $\varepsilon > 0$  sufficiently small we have

$$\mathcal{E}_0(u) - \mathcal{E}_0(\phi) \geq C \left\| e^{i\beta(u)} u(\cdot + \tau(u)) - \phi \right\|_X^2 \geq C \rho(u, \phi)^2,$$

for some constant  $C > 0$ , which completes the proof.  $\square$

Equipped with the coercivity estimate in Proposition 4.4 above, we now establish orbital stability of  $\phi_0$  with respect to complex-valued,  $T$ -antiperiodic perturbations.

*Proof of Theorem 4.1.* Let  $\varepsilon_0 > 0$  be such that Proposition 4.4 holds, and let  $v \in X$  satisfy  $\rho(\phi_0 + v) \leq \varepsilon$  for some  $0 < \varepsilon < \varepsilon_0$ . By possibly replacing  $v$  with  $e^{i\beta} v(\cdot + x_0)$  for some  $\beta, x_0 \in \mathbb{R}$ , we may assume without loss of generality that  $\|v\|_X < \varepsilon$ . Since  $\phi_0$  is a critical point of  $\mathcal{E}_0$ , Taylor's theorem implies that  $\mathcal{E}_0(\phi_0 + v) - \mathcal{E}_0(\phi_0) \leq C\varepsilon^2$  for some constant  $C > 0$ . Furthermore, notice that if  $\phi_0 + v \in \Sigma_0$ , then the unique solution  $u(\cdot, t)$  of (4.1) with  $u(\cdot, 0) = \phi_0 + v$  remains in  $\Sigma_0$  so long as it exists. Since  $\mathcal{E}_0(u(\cdot, t)) = \mathcal{E}_0(u(\cdot, 0)) = \mathcal{E}_0(\phi_0 + v)$  independently of  $t$ , we have by the coercivity estimate in Proposition 4.4 that

$$C^{-1} \rho(u(\cdot, t), \phi_0)^2 \leq \mathcal{E}_0(\phi_0 + v) - \mathcal{E}_0(\phi_0) \leq C\varepsilon^2$$

for some constant  $C > 0$  and all  $t \geq 0$ , establishing the orbital stability of  $\phi_0$  to such perturbations.

In the case that  $\phi_0 + v \notin \Sigma_0$  but  $\|v\|_X \leq \varepsilon$  and  $N(\phi_0 + v) = 0$ , we utilize the nondegeneracy of the constraint set to establish stability. Specifically, recall that from the discussion following the proof of Proposition 3.1 that the condition (4.3) implies that the mapping

$$(c, \mu) \mapsto (N(\phi(\cdot; c, \mu)), Q(\phi(\cdot; c, \mu)))$$

is a period-preserving diffeomorphism from a neighborhood of  $(c, \mu) = (c_0, \mu_0)$  onto a neighborhood of  $(N, Q) = (0, \mu_0)$ . We may thus find a number  $\tilde{\mu} \in \mathbb{R}$  with  $\tilde{\mu} = \mathcal{O}(\varepsilon)$  such that  $\phi_\varepsilon(\cdot; \mu_0 + \tilde{\mu})$  is a real-valued  $T$ -antiperiodic standing wave of FNLS (1.1) satisfying  $Q(\phi_\varepsilon(\cdot; \mu_0 + \tilde{\mu})) = Q(\phi_0 + v)$ . Defining  $\mathcal{E}_\varepsilon(u) = \mathcal{E}(u) + \omega(\mu + \tilde{\mu})Q(u)$ , we may furthermore assume that  $\phi_\varepsilon$  minimizes  $\mathcal{E}_\varepsilon$  subject to the constraint that  $Q(u) = Q(\phi_0 + v)$  and  $N(u) = 0$ . From the proof of Proposition 4.4, we now have

$$\mathcal{E}_\varepsilon(u) - \mathcal{E}_\varepsilon(\phi_\varepsilon) \geq C \rho(u, \phi_\varepsilon)^2$$

so long as  $\rho(u, \phi_\varepsilon)$  sufficiently small and, since  $\phi_\varepsilon$  is a critical point of  $\mathcal{E}_\varepsilon$ , we also have by the triangle inequality that

$$\mathcal{E}_\varepsilon(u(\cdot, t)) - \mathcal{E}_\varepsilon(\phi_\varepsilon) = \mathcal{E}_\varepsilon(\phi_0 + v) - \mathcal{E}_\varepsilon(\phi_\varepsilon) \leq C\varepsilon^2$$

for all  $t \geq 0$ . Again using the triangle inequality, we finally have

$$\begin{aligned} \rho(u(\cdot, t), \phi_0)^2 &\leq C (\rho(u(\cdot, t), \phi_\varepsilon)^2 + \rho(\phi_\varepsilon, \phi_0)^2) \\ &\leq C (\mathcal{E}_\varepsilon(u) - \mathcal{E}_\varepsilon(\phi_\varepsilon)) + C \|\phi_\varepsilon - \phi_0\|_X^2 = 2C\varepsilon^2 \end{aligned}$$

for all  $t \geq 0$ , implying that  $\phi_0$  is orbitally stable to small perturbations that “slightly” change  $Q$  yet preserve  $N$ , thus establishing Theorem 4.1.  $\square$

## 5 Analysis of the Focusing Case

As described in the introduction, following Gallay & Haragus[20] we choose not to give full attention to periodic standing waves in the focusing case ( $\gamma = +1$ ). In large part, this choice is due to the observation of Rowlands [48] that, at least in the local  $\alpha = 2$  case with  $\sigma = 1$ , all such waves are modulationally unstable; see also the recent works [15, 28]. While it is not immediately clear that Rowlands’ result extends to the nonlocal case  $\alpha \in (0, 2)$ , it seems reasonable to expect. Thus, while we may be able to establish the stability of antiperiodic waves in this case to a restricted class of perturbations, these waves are expected to be unstable to more general periodic perturbations. Nevertheless, the theory developed in the previous sections is still able to establish the nondegeneracy of all the waves we construct here, irrespective of whether such waves are constrained energy minimizers, an observation we believe is worth discussing in some detail.

To motivate the existence of such waves, observe that in the local case  $\alpha = 2$  with  $\gamma = +1$  elementary phase plane analysis reveals that the focusing profile equation

$$(5.1) \quad \Lambda^\alpha \phi + \omega \phi + ic\phi' - |\phi|^{2\sigma} \phi = 0, \quad \omega, c \in \mathbb{R}$$

for  $c = 0$  yields several families of real-valued, bounded periodic solutions. Indeed, integrating (5.1) implies such waves can be reduced to quadrature via

$$\frac{1}{2} (\phi')^2 = H - V(\phi; \omega),$$

where

$$V(\phi; \omega) = -\frac{\omega}{2} \phi^2 + \frac{1}{2\sigma + 2} \phi^{2\sigma+2};$$

see Figure 2. When  $\omega > 0$ , there exist sign-definite periodic solutions as well as sign changing antiperiodic solutions, with these classes of solutions being separated in phase space by a separatrix corresponding to the unique (up to symmetries) solitary standing wave: mark that no such homoclinic structures exist in the defocusing case. Furthermore, when  $\omega < 0$  all periodic solutions are in fact antiperiodic and there exist no other real-valued, nonconstant standing structures. The nonlinear stability of such periodic structures in the local case  $\alpha = 2$  has been studied in [2, 20, 28]. As in the previous sections, our goal here is to extend this local analysis to the genuinely nonlocal case.

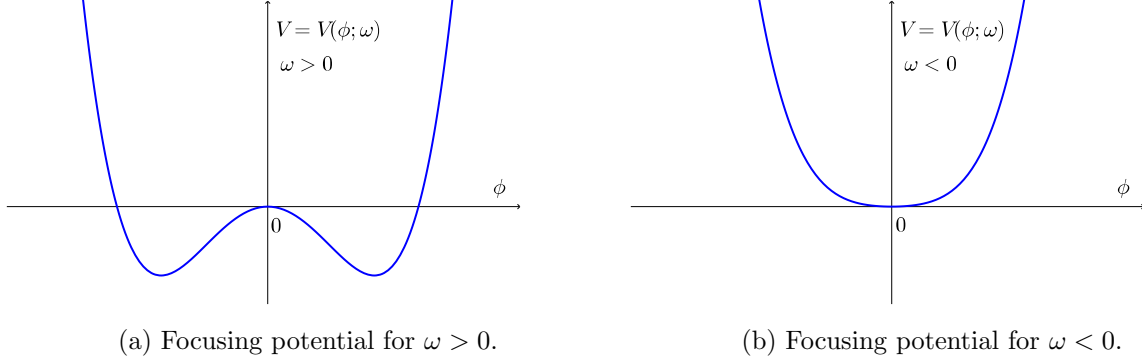


Figure 2: The effective potential  $V(\phi; \omega) = -\frac{\omega}{2}\phi^2 + \frac{1}{2\sigma+2}\phi^{2\sigma+2}$  for the focusing NLS.

When  $\alpha \in (1, 2)$ , it is natural to expect the existence of both antiperiodic and sign-definite waves for the focusing fNLS. Mimicking the analysis from the defocusing case, we concentrate here on the existence, nondegeneracy, and stability of antiperiodic standing wave solutions: see Remark 5.4 for discussion regarding the sign-definite solutions. The existence of real-valued, antiperiodic solutions of (5.1) may be established by a similar method as in Section 2, though with a different functional setup. Observe that  $T$ -antiperiodic solutions of (5.1) are critical points of the Lagrangian functional

$$H_a^{\alpha/2}(0, T) \ni u \mapsto K(u) - P(u) + \omega Q(u),$$

where  $K$ ,  $P$ , and  $Q$  are defined as in Section 2: note the sign difference on the potential energy term  $P$  between here and the defocusing case. As the Hamiltonian  $K - P$  is not sign-definite in this case, rather than constructing critical points by minimizing  $K - P$  subject to fixed  $Q$ , producing constrained energy minimizers, we seek to minimize  $K + \omega Q$  for a given  $\omega$  subject to fixed potential energy.

**Proposition 5.1.** *Let  $\alpha \in (1, 2)$ ,  $\omega \in \mathbb{R}$  and  $T, \sigma > 0$  be fixed in the focusing ( $\gamma = +1$ ) fNLS (1.1). For each  $|\omega| < (\frac{\pi}{T})^\alpha$  and  $P_0 > 0$  there exists a real-valued, even  $\phi \in H_a^{\alpha/2}(0, T)$  with  $P(\phi) = P_0$  such that  $\phi$  is strictly decreasing on  $(0, T)$  and solves (5.1) with  $c = 0$  in the sense of distributions. The function  $\phi(\cdot; \omega, P_0)$  depends on  $\omega$  and  $P_0$  in a  $C^1$  manner and  $\phi \in H_a^\infty(0, T)$ . Furthermore,  $\phi$  minimizes the Lagrangian functional*

$$\mathcal{E}(u; \omega) := K(u) - P(u) + \omega Q(u)$$

subject to the constraint  $P(u) \equiv P_0$ .

To prove the above proposition, the following Poincaré inequality will be useful.

**Lemma 5.2** (Poincaré). *For all  $\alpha > 0$ , we have  $Q(u) \leq (\frac{T}{\pi})^\alpha K(u)$  for all  $u \in H_a^{\alpha/2}(0, T)$ .*

*Proof.* By Parseval, we have

$$Q(u) = \frac{1}{2} \|\hat{u}\|_{\ell^2}^2 \leq \frac{1}{2} \left( \frac{T}{\pi} \right)^\alpha \left\| \left| \frac{\pi n}{T} \right|^{\alpha/2} \hat{u}(n) \right\|_{\ell_n^2}^2 = \left( \frac{T}{\pi} \right)^\alpha K(u),$$

as claimed. □



*Proof of Proposition 5.1.* For each  $\omega \in \mathbb{R} \setminus \{0\}$ , consider the functional

$$R_\omega(u) := K(u) + \omega Q(u)$$

and for each  $P_0 > 0$  define the constraint set

$$\mathcal{B}_{P_0} := \left\{ u \in H_a^{\alpha/2}(0, T) : P(u) = P_0 \right\}.$$

Observe that by Lemma 5.2 we have

$$R_\omega(u) = K(u) + \omega Q(u) \geq \left( \left( \frac{\pi}{T} \right)^\alpha + \omega \right) Q(u) > 0$$

and hence that  $R_\omega$  is bounded below on  $\mathcal{B}_{P_0}$  provided  $\omega > -\left(\frac{\pi}{T}\right)^\alpha$ . Consequently, for such  $\omega$  the number  $\lambda := \inf_{u \in \mathcal{A}} R_\omega(u)$  is well-defined, so there exists a minimizing sequence  $\{u_k\}_{k=1}^\infty \subset \mathcal{B}_{P_0}$  such that  $R_\omega(u_k) \rightarrow \lambda$  as  $k \rightarrow \infty$ . Moreover, using Lemma 5.2 again we find that

$$R_\omega(u) \geq \left( 1 - \left( \frac{T}{\pi} \right)^\alpha |\omega| \right) K(u)$$

and hence, using Lemma 5.2 again, for  $|\omega| < \left(\frac{\pi}{T}\right)^\alpha$  we have

$$\frac{1}{2} \|u_k\|_{H_a^{\alpha/2}(0, T)}^2 \leq \left( 1 + \left( \frac{T}{\pi} \right)^\alpha \right) K(u_k) \leq \frac{1 + \left(\frac{T}{\pi}\right)^\alpha}{1 - \left(\frac{T}{\pi}\right)^\alpha |\omega|} R_\omega(u_k).$$

It follows that for such  $\omega$  the sequence  $\{u_k\}_{k=1}^\infty$  is bounded in  $H_a^{\alpha/2}(0, T)$ , thus by Banach-Alaoglu there exists a further subsequence  $\{u_{k_j}\}_{j=1}^\infty$  converging weakly in  $H_a^{\alpha/2}(0, T)$  and strongly (in norm) in  $L^2(0, T)$  to some function  $\phi \in H_a^{\alpha/2}(0, T)$ . Since  $\alpha > 1$  the space  $H_a^{\alpha/2}(0, T)$  is compactly embedded into both  $L^2(0, T)$  and  $L^{2\sigma+2}(0, T)$ , and hence that

$$Q(\phi) = \lim_{j \rightarrow \infty} Q(u_{k_j}) \quad \text{and} \quad P(\phi) = \lim_{j \rightarrow \infty} P(u_{k_j}) = P_0.$$

Consequently,  $\phi \in \mathcal{B}_{P_0}$ . Finally, since  $K$  is weakly lower-semicontinuous on  $H_a^{\alpha/2}(0, T)$ , we have

$$\liminf_{j \rightarrow \infty} R_\omega(u_{k_j}) \geq K(\phi) + \omega Q(\phi) = R_\omega(\phi)$$

and thus

$$\lambda \leq R_\omega(\phi) \leq \liminf_{j \rightarrow \infty} R_\omega(u_{k_j}) = \lim_{j \rightarrow \infty} R_\omega(u_{k_j}) = \lambda.$$

It follows that for each  $|\omega| < \left(\frac{\pi}{T}\right)^\alpha$  and  $P_0 > 0$ , there exists a nontrivial  $\phi \in \mathcal{B}_{P_0}$  such that

$$R_\omega(\phi) = \inf_{u \in \mathcal{B}_{P_0}} R_\omega(u).$$

By Lagrange multipliers, there exists  $\eta \in \mathbb{R}$  such that

$$(5.2) \quad \Lambda^\alpha \phi + \omega \phi + \eta |\phi|^{2\sigma} \phi = 0.$$

Clearly  $\phi$  and  $\eta$  depend on  $\omega$  and  $P_0$  in a  $C^1$  manner.

Continuing, observe that multiplying (5.2) by  $\frac{1}{2}\bar{\phi}$  and integrating yields

$$R_\omega(\phi) + (\sigma + 1)\eta P(\phi) = 0,$$

and hence it must be that  $\eta < 0$ . Rescaling  $\phi \mapsto |\eta|^{-1/2\sigma}\phi$  we find that  $\phi$  solves the focusing fNLS profile equation (5.1) with  $c = 0$ . Since  $R_\omega(a\phi) = a^2 R_\omega(\phi)$ , it follows that  $\phi$  is a constrained minimizer of  $R_\omega$  subject to fixed  $P \equiv |\eta|^{-(1+1/\sigma)}P_0$ . Consequently, for every  $|\omega| < \left(\frac{\pi}{T}\right)^\alpha$  there exists a function  $\phi$  that minimizes  $R_\omega$  subject to a suitably fixed potential energy  $P \equiv P_0$ . Note that, as in the defocusing case, we may without loss of generality take  $\phi$  to be real-valued and even since  $P$  and  $Q$  are preserved under taking absolute value and symmetric decreasing rearrangement, while  $K$  does not increase under these operations: see Lemma A.1. Furthermore, following arguments as in Section 2 we find  $\phi \in H_a^\infty(0, T)$ . Finally, since

$$\mathcal{E}(\phi) = \inf_{P(u)=P(\phi)} (K(u) + \omega Q(u) + P(u)) = \inf_{P(u)=P(\phi)} \mathcal{E}(u),$$

it follows that the Lagrangian functional  $\mathcal{E}(\cdot; \omega)$  is also minimized by  $\phi$  subject to fixed  $P \equiv P(\phi)$ , completing the proof.  $\square$

Observe that the antiperiodic standing waves constructed in Proposition 5.1 need not be constrained minimizers of the Hamiltonian energy  $K - P$ . Indeed, using the second derivative test for constrained extrema in this case implies that

$$\delta^2 \mathcal{E}(\phi) \Big|_{\{\delta P(\phi)\}^\perp} \geq 0,$$

where  $\delta P(\phi) = |\phi|^{2\sigma}\phi$  is real-valued. Consequently, the operator  $\delta^2 \mathcal{E}(\phi)$  has at most one negative eigenvalue when acting on  $L_a^2(0, T)$ . More precisely, decomposing into real and imaginary parts, the fact that  $\delta P(\phi)$  is real-valued implies that

$$n_-(L_+) \leq 1 \quad \text{and} \quad L_- \geq 0.$$

Following the spirit of the arguments in Proposition 3.1 above, we can establish nondegeneracy of both  $L_\pm$  in this focusing case, as well determine the Morse index of  $L_+$ .

**Proposition 5.3.** *Let  $\alpha \in (1, 2)$  and  $\sigma > 0$  in the focusing ( $\gamma = +1$ ) fNLS (1.1). If  $\phi(\cdot; \omega, P_0) \in H_a^{\alpha/2}(0, T)$  is a local minimizer of  $R_\omega$  subject to fixed potential energy  $P \equiv P_0$ , then the associated Hessian operator acting on  $L_a^2(0, T)$  is nondegenerate, i.e.*

$$\ker(\delta^2 \mathcal{E}(\phi)) = \text{span}\{\phi', i\phi\}$$

and  $n_-(\delta^2 \mathcal{E}(\phi)) = 1$ . Specifically, the operators  $L_\pm$  are nondegenerate acting on  $L_a^2(0, T)$  with

$$\ker(L_+) = \text{span}\{\phi'\}, \quad \text{and} \quad \ker(L_-) = \text{span}\{\phi\}$$

and, further,  $n_-(L_+) = 1$  and  $n_-(L_-) = 0$ .

*Proof.* Since  $L_- \phi = 0$  and  $\phi$  is even, it follows from Theorem 3.9 that  $\phi$  is the ground state of  $L_-$  on the even sector  $L_{a,\text{even}}^2(0, T)$ , and hence  $\lambda = 0$  is a simple eigenvalue of  $L_-$  restricted to the even sector. If  $L_-$  were degenerate there would exist a function  $\psi \in L_{a,\text{odd}}^2(0, T)$  such that  $L_- \psi = 0$ .

But the ground state theory Theorem 3.9 again would imply that  $\psi$  may be taken to be strictly positive on  $(0, T)$ , and hence cannot be orthogonal to the function  $\phi' \in \text{range}(L_-)$ , contradicting the Fredholm alternative. Consequently,  $L_-$  is nondegenerate on  $L_a^2(0, T)$  and  $L_- \geq 0$ , as claimed.

Next, we claim that  $L_+$  is nondegenerate. First, note that since  $L_+\phi' = 0$  and  $\phi'$  is strictly negative on  $(0, T)$ , it follows by the ground state theory Theorem 3.9 that  $\phi'$  is the ground state eigenfunction<sup>9</sup> of  $L_+$  on the odd sector  $L_{a,\text{odd}}^2(0, T)$ . Hence,  $\lambda = 0$  is a simple eigenvalue of  $L_+$  restricted to the odd sector. Since  $\phi$  is strictly decreasing on  $(0, T)$  and  $\gamma = +1$  here, Proposition 3.10(ii) implies that the ground state eigenvalue of  $L_+$  on  $L_a^2(0, T)$  is nonpositive and has at least one even eigenfunction. To show nondegeneracy, we first show that  $L_+$  has exactly one negative eigenvalue. If  $L_+ \geq 0$ , then by the above discussion  $\lambda = 0$  must be the ground state eigenvalue for  $L_+$  on the even sector, and hence there exists a  $\psi \in L_{a,\text{even}}^2(0, T)$  with  $\psi(x) > 0$  on  $(-T/2, T/2)$  such that  $L_+\psi = 0$ . This, however, contradicts the Fredholm alternative since such a  $\psi$  could not be orthogonal to the function  $\phi$ , which is also strictly positive on  $(-T/2, T/2)$  lies in the range of  $L_+$  since, differentiating (5.1) with  $c = 0$  with respect to  $\omega$  yields

$$L_+ \frac{\partial \phi}{\partial \omega} = -\phi.$$

Thus, the even ground state eigenvalue of  $L_+$  must be strictly negative, establishing that  $n_-(L_+) = 1$  as claimed.

To conclude nondegeneracy of  $L_+$ , note now that  $\lambda = 0$  must be the second eigenvalue of  $L_+$ . If  $L_+$  were degenerate, the simplicity of  $\lambda = 0$  on the odd sector  $L_{a,\text{odd}}^2(0, T)$  implies that there must exist a function  $\zeta \in L_{a,\text{even}}^2(0, T)$  such that  $L_+\zeta = 0$ . By the oscillation theory Lemma 3.11, it follows then that  $\zeta$  can change signs at most twice on  $(-T/2, T/2)$ . Clearly  $\zeta$  cannot be sign-definite on  $(-T/2, T/2)$  since then  $\zeta$  would not be orthogonal to  $\phi$ , contradicting again the Fredholm alternative. Since  $\zeta$  is even, it may be normalized so there exists an  $x_0 \in (0, T/2)$  such that  $\zeta(x) > 0$  on  $(-x_0, x_0)$  and  $\zeta(x) < 0$  for  $x \in (-T/2, -x_0) \cup (x_0, T/2)$ . However, one can easily check that the function

$$\phi(x) (\phi(x)^{2\sigma} - \phi(x_0)^{2\sigma}) \in \text{range}(L_+)$$

has the same nodal pattern of  $\zeta$ , again contradicting the Fredholm alternative. It follows that  $\ker(L_+|_{L_{a,\text{even}}^2(0, T)}) = \{0\}$ , and hence  $\ker(L_+|_{L_a^2(0, T)}) = \text{span}\{\phi'\}$ , as claimed.  $\square$

*Remark 5.4.* Although we do not pursue full arguments here, it is worth noting that the nondegeneracy of the sign definite solutions of the focusing fNLS with  $\omega < 0$  will follow directly from the analysis of Hur and Johnson in [42]. In that case, we would consider positive,  $T$ -periodic solutions  $\phi$  of (5.1) with  $c = 0$  that would be even, smooth, and strictly decreasing on  $(0, T/2)$ . Such waves could be constructed as critical points of  $R_\omega$  above considered now as acting on  $L_{\text{per}}^2(0, T)$ . As above, these waves would have  $L_- \geq 0$  thanks to the ground state characterization Theorem 3.9, and nondegeneracy of  $L_-$  with respect to  $T$ -periodic perturbations would follow as above. For  $L_+$  one would have  $n_-(L_+) = 1$  and nondegeneracy would follow directly from the arguments in [42, Section 3]. The stability of these waves should then follow the same characterization as described in the forthcoming analysis, depending on the sign of the Jacobian  $\frac{\partial Q}{\partial \omega}(\phi)$ .

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<sup>9</sup>Recall eigenfunctions corresponding to distinct eigenvalues of  $L_+$  must be orthogonal, hence sign-definite eigenfunctions correspond to the ground state eigenvalues.

Recapitulating, for the focusing fNLS we have constructed a four-parameter family<sup>10</sup> of real-valued, even antiperiodic solutions whose linearizations are necessarily nondegenerate. Since such solutions were not constructed as constrained minimizers of the Hamiltonian energy, however, the nonlinear stability of these waves is not guaranteed as it was for the defocusing analysis. Rather, as is common with the local case, the stability depends on the sign of the quantity  $\frac{\partial Q}{\partial \omega}$ . As a preliminary step in this direction, we show that if  $\frac{\partial Q}{\partial \omega}$  is positive at the underlying wave, then this wave is necessarily a constrained minimizer of the Hamiltonian energy subject to fixed  $L^2$ -norm.

**Lemma 5.5.** *Let  $\phi = \phi(\cdot; \omega)$  be a real-valued, even  $T$ -antiperiodic solution of the focusing fNLS as constructed in Proposition 5.1. If  $\frac{\partial Q}{\partial \omega}(\phi) > 0$ , then  $\phi$  minimizes the Lagrangian functional  $\mathcal{E}$  subject to fixed  $L^2$ -norm, i.e.*

$$\delta^2 \mathcal{E}(\phi)|_{\{\delta Q(\phi)\}^\perp} \geq 0,$$

*In particular, for such a wave we necessarily have  $L_+|_{\{\phi\}^\perp} \geq 0$ .*

*Proof.* We use a recent result [29, Lemma 1] regarding the non-negativity of a self-adjoint operator having a spectral gap and exactly one negative eigenvalue per the above discussion. We readily verify the hypotheses of their result, as  $L_+$  is self-adjoint on the Hilbert space  $H_a^{\alpha/2}(0, T)$ , and  $L_+$  has exactly one negative eigenvalue, which is the (simple) ground state. Lastly, observe that  $\phi \in \ker(L_+)^{\perp}$ , with  $L_+^{-1}\phi = -\frac{\partial \phi}{\partial \omega}$ . Then

$$\langle L_+^{-1}\phi, \phi \rangle = -\left\langle \frac{\partial \phi}{\partial \omega}, \phi \right\rangle = -\frac{\partial Q}{\partial \omega}(\phi(\cdot; \omega)) < 0,$$

hence by [29, Lemma 1] it follows that  $L_+|_{\{\phi\}^\perp} \geq 0$ . Since  $L_- \geq 0$  a-priori, we conclude that  $\delta^2 \mathcal{E}(\phi)|_{\{\phi\}^\perp} \geq 0$ .  $\square$

For the waves constructed in Proposition 5.1, we have now established that  $\delta^2 \mathcal{E}(\phi)$  is non-degenerate, with  $\ker(\delta^2 \mathcal{E}(\phi)) = \text{span}\{\phi', i\phi\}$  and that, by Lemma 5.5, that such waves are constrained minimizers of the Hamiltonian energy subject to fixed  $L^2$  norm provided that  $\frac{\partial Q}{\partial \omega}(\phi) > 0$ . As in Section 4, such conditions are sufficient to conclude the nonlinear orbital stability of a given wave with respect to  $T$ -antiperiodic perturbations that slightly change  $Q$ . Indeed, the proof of coercivity is very similar to that of Proposition 4.4, and the stability result ensues with very few modifications<sup>11</sup> to the proof Theorem 4.1. Due to the similarity, we only state the result.

**Theorem 5.6.** *Suppose  $\alpha \in (1, 2]$ ,  $\sigma > 0$  and that  $X$  is a suitable subspace of  $H_a^{\alpha/2}([0, T]; \mathbb{C})$  where the Cauchy problem associated with*

$$iu_t - \Lambda^\alpha u + |u|^{2\sigma} u = 0$$

*is well-posed and the functionals  $\mathcal{H}, Q, N : X \rightarrow \mathbb{R}$  are smooth. Further, let  $\phi_0 \in H_a^{\alpha/2}(0, T)$  be a real-valued  $T$ -antiperiodic standing solution of (5.1) as constructed in Proposition 5.1 such that*

$$\frac{\partial Q}{\partial \omega}(\phi_0) > 0.$$

<sup>10</sup>We can parameterize by  $\omega$  and  $P_0$ , together with the continuous Lie-point symmetries coming from translational and gauge invariances of the governing evolution equation.

<sup>11</sup>In fact, the required analysis follows the stability theory for the solitary wave case [51].

Then for all  $\varepsilon > 0$  sufficiently small there exists a constant  $C = C(\varepsilon)$  such that if  $v \in X$  with  $\|v\|_X \leq \varepsilon$  and  $u(\cdot, t)$  is a local in time solution of

$$iu_t - \omega u - \Lambda^\alpha u + |u|^{2\sigma} u = 0$$

with initial data  $u(\cdot, 0) = \phi_0 + v$ , then  $u(\cdot, t)$  can be continued to a solution for all  $t > 0$  and

$$\sup_{t>0} \inf_{(\beta, x_0) \in \mathbb{R}^2} \left\| u(\cdot, t) - e^{i\beta} \phi_0(\cdot - x_0) \right\|_X \leq C \|v\|_X.$$

We remark that the criterion  $\frac{\partial Q}{\partial \omega}(\phi_0) > 0$  in Theorem 5.6 is a sufficient condition for orbital stability in the focusing case, compensating for the fact that  $\phi$  was not constructed as a minimizer of  $\mathcal{E}$  with fixed  $Q$ : see Lemma 5.5. Conversely,  $\phi_0$  can be shown to be spectrally unstable if  $\frac{\partial Q}{\partial \omega}(\phi_0) < 0$ .

**Proposition 5.7.** *Let  $\phi_0$  be as in Theorem 5.6. If  $\frac{\partial Q}{\partial \omega}(\phi_0) < 0$ , then  $\phi_0$  is spectrally unstable.*

The proof follows the now standard Vakhitov-Kolokolov projection method, and the interested reader is referred to [49, Section 4.1] for details.

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## A Antiperiodic Rearrangement Inequalities

In this section, we establish results pertaining to symmetric decreasing rearrangements of  $T$ -antiperiodic functions and their consequences. Given a function  $f \in L^2_{\text{per}}([0, 2T]; \mathbb{R}) \cup C^0(\mathbb{R})$  we will end up utilizing *four separate equimeasurable rearrangements* of  $f$ , which we will describe below.

Throughout, we denote by  $m$  the Lebesgue measure on  $\mathbb{R}/2T\mathbb{Z}$ . Given a continuous  $2T$ -periodic function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define the  $2T$ -periodic symmetric decreasing rearrangement  $f^{*2T}$  of  $f$  on  $(-T, T)$  by

$$f^{*2T}(x) = \inf \{t : m(\{z \in (-T, T) : f(z) > t\}) \leq 2|x|\} \quad \text{for } x \in [-T, T]$$

and note that  $f^{*2T}$  is even, nonincreasing on  $(0, T)$ , and satisfies  $f^{*2T}(0) = \max_{x \in \mathbb{R}} f(x)$  and  $f^{*2T}(T) = \min_{x \in \mathbb{R}} f(x)$ . Similarly, we define the  $2T$ -periodic rearrangement  $f^{\#2T}(x)$  by

$$f^{\#2T}(x) = f^{*2T}(x - T/2)$$

and note that  $f^{\#}$  is even about  $x = T/2$ , nondecreasing on  $(-T/2, T/2)$  and satisfies  $f^{\#2T}(T/2) = \max_{x \in \mathbb{R}} f(x)$  and  $f^{\#2T}(-T/2) = \min_{x \in \mathbb{R}} f(x)$ . Both  $f^{*2T}$  and  $f^{\#2T}$  have the same distribution functions as  $f$  on  $(-T, T)$  so that, in particular,

$$\|f\|_{L^p(-T, T)} = \|f^{*2T}\|_{L^p(-T, T)} = \|f^{\#2T}\|_{L^p(-T, T)}$$

for all  $f \in L_{\text{per}}^p(0, T)$  and  $1 \leq p \leq \infty$ . Of special interest here is that if  $f$  is  $T$ -antiperiodic, then  $f^{*2T}$  is an even,  $T$ -antiperiodic function on  $\mathbb{R}$  while  $f^{\#2T}$  is an odd,  $T$ -antiperiodic function on  $\mathbb{R}$ .

Our first result is an analogue of the classical Pólya-Szegő inequality, which states that the kinetic energy is nonincreasing under symmetric decreasing rearrangement.

**Lemma A.1** (Pólya-Szegő). *For all  $\alpha \in (1, 2)$  and  $f \in H_{\text{per}}^{\alpha/2}([0, 2T]; \mathbb{R})$ , we have*

$$\int_{-T}^T \left| \Lambda^{\alpha/2} f^{*2T} \right|^2 dx = \int_{-T}^T \left| \Lambda^{\alpha/2} f^{\#2T} \right|^2 dx \leq \int_{-T}^T \left| \Lambda^{\alpha/2} f \right|^2 dx.$$

*In particular, if such an  $f$  is  $T$ -antiperiodic, then*

$$\int_{-T/2}^{T/2} \left| \Lambda^{\alpha/2} f^{*2T} \right|^2 dx = \int_{-T/2}^{T/2} \left| \Lambda^{\alpha/2} f^{\#2T} \right|^2 dx \leq \int_{-T/2}^{T/2} \left| \Lambda^{\alpha/2} f \right|^2 dx.$$

*Proof.* Given  $f \in H_{\text{a}}^{\alpha/2}([0, T]; \mathbb{R})$ , observe that for all  $t > 0$  we have

$$\langle f, e^{-\Lambda^{\alpha} t} f \rangle = \int_{-T}^T \int_{-T}^T f(x) K_p(x - y, t) f(y) dx dy,$$

where here  $K_p(x, t)$  is the  $2T$ -periodic integral kernel associated to the semigroup  $e^{-\Lambda^{\alpha} t}$  defined in (3.7). By Lemma 3.2,  $K_p(\cdot, t) = K_p^{*2T}(\cdot, t)$  for all  $t > 0$  and hence the Bernstein-Taylor Theorem [3, Theorem 2] we have

$$\int_{-T}^T \int_{-T}^T f(x) K_p(x - y, t) f(y) dx dy \leq \int_{-T}^T \int_{-T}^T f^{*2T}(x) K_p(x - y, t) f^{*2T}(y) dx dy$$

so that

$$\langle f, e^{-\Lambda^{\alpha} t} f \rangle \leq \langle f^{*2T}, e^{-\Lambda^{\alpha} t} f^{*2T} \rangle$$

for all  $f \in H_{\text{a}}^{\alpha/2}([0, T]; \mathbb{R})$  and  $t > 0$ . Since  $f$  and  $f^{*2T}$  are equimeasurable, it follows that

$$\frac{\langle f, e^{-\Lambda^{\alpha} t} f \rangle - \|f\|_{L^2(-T, T)}^2}{t} \leq \frac{\langle f^{*2T}, e^{-\Lambda^{\alpha} t} f^{*2T} \rangle - \|f^{*2T}\|_{L^2(-T, T)}^2}{t}$$

for all  $t > 0$ . Taking  $t \rightarrow 0^+$  yields the desired result for the rearrangement  $f^{*2T}$ . The corresponding result for  $f^{\#2T}$  and the restriction to  $T$ -antiperiodic functions now follows trivially.  $\square$

Next, we complement the above result by considering the effect of the above rearrangements on linear potentials.

**Lemma A.2.** *Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be an even, smooth and  $T$ -periodic potential.*

*(i) If  $V(x)$  is nonincreasing on  $(0, T/2)$ , then*

$$\int_{-T/2}^{T/2} V(x) f^2(x) dx \geq \int_{-T/2}^{T/2} V(x) \left( f^{\#2T} \right)^2(x) dx$$

*for all continuous  $f \in L_{\text{a}}^2([0, T]; \mathbb{R})$ .*

(ii) If  $V(x)$  is nondecreasing on  $(0, T/2)$ , then

$$\int_{-T/2}^{T/2} V(x) f^2(x) dx \geq \int_{-T/2}^{T/2} V(x) (f^{*2T})^2(x) dx$$

for all continuous  $f \in L^2_{\text{a}}([0, T]; \mathbb{R})$ .

*Proof.* We begin by proving (i). Notice by the hypothesis on  $V$ , the function  $(-V(x))$  is even about  $x = T/2$  and is nonincreasing on  $(T/2, T)$ . By the Riesz inequality [39, Section 3.4] we thus have

$$\int_0^T (-V(x)) f^2(x) dx \leq \int_0^T (-V(x)) (f^2)^{\#T}(x) dx,$$

where here  $(f^2)^{\#T}$  denotes the  $T$ -periodic rearrangement of the  $T$ -periodic function  $f^2$  taken to be even about  $x = T/2$  and nonincreasing on  $(T/2, T)$ . Since antiperiodicity of  $f$  implies

$$(f^2)^{\#T}(x) = (f^{\#2T})^2(x) \quad \forall x \in (0, T),$$

the estimate in (i) follows.

Similarly, if  $V$  satisfies the hypotheses of (ii), then the Riesz inequality again gives

$$\int_0^T (-V(x)) f^2(x) dx \leq \int_0^T (-V(x)) (f^2)^{*T}(x) dx,$$

where here  $(f^2)^{*T}$  denotes the  $T$ -periodic rearrangement of the  $T$ -periodic function  $f^2$  taken to be even about  $x = 0$  and nonincreasing on  $(0, T/2)$ . Antiperiodicity of  $f$  again implies that

$$(f^2)^{*T}(x) = (f^{*2T})^2(x) \quad \forall x \in (0, T/2),$$

the estimate in (ii) follows.  $\square$

We now come to the main result of this appendix, providing an ordering between the even and odd ground state antiperiodic eigenvalues of a periodic Schrödinger operator  $L = -\Lambda^\alpha + V$  in terms of the monotonicity properties of the potential  $V$ .

*Proof of Proposition 3.10:* First, assume that  $V(x)$  satisfies the hypothesis of (i) and suppose that  $\psi$  is an eigenfunction associated to the ground state eigenvalue of  $L$  acting on  $L^2_{\text{a,even}}(0, T)$ , normalized to be real-valued and  $\|\psi\|_{L^2(0, T)} = 1$ . Then by Lemma A.1 and Lemma A.2

$$\begin{aligned} \min \sigma \left( L|_{L^2_{\text{a,even}}(0, T)} \right) &= \int_0^T \left| \Lambda^{\alpha/2} \psi \right|^2 dx + \int_0^T V(x) \psi(x)^2 dx \\ &\geq \int_0^T \left| \Lambda^{\alpha/2} \psi^{\#2T} \right|^2 dx + \int_0^T V(x) \left( \psi^{\#2T} \right)^2(x) dx \\ &\geq \min \sigma \left( L|_{L^2_{\text{a,odd}}(0, T)} \right), \end{aligned}$$

where the last inequality is justified since  $\|\psi^{\#2T}\|_{L^2(0, T)} = 1$ . This verifies (i). A similar proof establishes the ordering in (ii).

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